Fourier Transform
Model multiplicity → count / numbers.

Model opposite → I have $5

How much should I add to make $0? Ans: $-5

Numbers & neg. numbers are all abstractions of reality

Now, How to model rotation?

We have \( \cos \theta \) and \( \sin \theta \). Of course we need to express their relation in some way. We cannot just say \( \cos \theta + \sin \theta \times \)

or \( \left< \cos \theta, \sin \theta \right> \)

which still does not allow algebra.

OK, we can say \( \hat{y} \cos \theta + \hat{y} \sin \theta \)

But we don't have to say \( \hat{y} \), we just need to make \( \hat{y} \) become 1 to \( \hat{x} \).

How? Well, one way is to say that \( \hat{y} \) should be \(-1\hat{x}\).

i.e., \( y^2 = -1 \) \(: y = \sqrt{-1} = j \)

\( \therefore \) Rotation is a vector of \( \cos \theta + j \sin \theta \)

Aha, this is exactly \( e^{j\theta} \Rightarrow \) models rotation.
Now, what would rotation at different freq. look like? Say we want to model every freq. in $N$ discrete steps.

Fourier observed each column is orthogonal.

$$f_0 \cdot f_1 = 0 \Rightarrow \text{you can immediately see because}$$

Same holds for any pair.
Vector Projection to Fourier Basis

\[ \mathbf{X} = [x[0], x[1], x[2], \ldots, x[N-1]] \]

Vector in \( N \) dimensional space.
Let's denote \( x[0] \) as \( x_0 \) from now on.

\[ \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
x_0 \\
x_1 \\
x_2 \\
\vdots \\
x_{N-1}
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
f_0 & f_1 & f_2 & \cdots & f_{N-1}
\end{bmatrix}
\begin{bmatrix}
z_0 \\
z_1 \\
z_2 \\
\vdots \\
z_{N-1}
\end{bmatrix} \]

Orthogonal Fourier Basis matrix

\[ z(m) = F^{-1}x \]

\[ z(m) = (F^*)^T x \]

\[ \begin{bmatrix}
z_0 \\
z_1 \\
z_2 \\
\vdots \\
z_{N-1}
\end{bmatrix} =
\begin{bmatrix}
f_0^* & \cdots & \cdots & \cdots & f_{N-1}^* \\
f_0^* & \cdots & \cdots & \cdots & f_{N-1}^* \\
f_0^* & \cdots & \cdots & \cdots & f_{N-1}^* \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
f_0^* & \cdots & \cdots & \cdots & f_{N-1}^*
\end{bmatrix}
\begin{bmatrix}
x_0 \\
x_1 \\
x_2 \\
\vdots \\
x_{N-1}
\end{bmatrix} \]

Discrete Fourier Transform (DFT)
Consider 3D case:

\[
\begin{bmatrix}
  z_0 \\
  z_1 \\
  z_2
\end{bmatrix} = \begin{bmatrix}
  f_0^* \\
  f_1^* \\
  f_2^*
\end{bmatrix}\begin{bmatrix}
  x_0 \\
  x_1 \\
  x_2
\end{bmatrix}
\]

\[
z_0 = e^{-j \frac{2\pi}{N} \cdot 0.0} \cdot x_0 + e^{-j \frac{2\pi}{N} \cdot 0.1} \cdot x_1 + e^{-j \frac{2\pi}{N} \cdot 0.2} \cdot x_2
\]

\[
z_1 = e^{-j \frac{2\pi}{N} \cdot 1.0} \cdot x_0 + e^{-j \frac{2\pi}{N} \cdot 1.1} \cdot x_1 + e^{-j \frac{2\pi}{N} \cdot 1.2} \cdot x_2
\]

\[
z_2 = e^{-j \frac{2\pi}{N} \cdot 2.0} \cdot x_0 + e^{-j \frac{2\pi}{N} \cdot 2.1} \cdot x_1 + e^{-j \frac{2\pi}{N} \cdot 2.2} \cdot x_2
\]

DFT:

\[
z(n) = \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} \cdot m \cdot n}
\]

IDFT:

\[
x[n] = \sum_{m=0}^{N-1} z(n) e^{j \frac{2\pi}{N} \cdot m \cdot n}
\]

Scaling by \( \frac{1}{\sqrt{N}} \)

Note that dot product = Projection only when the dot product is normalized.

Now, for DFT, observe that each column of \( F \) is not a unit vector, e.g., \([1 \ 1 \ 1]^T\).

The DFT needs to be scaled by the length of the vector \( 1/\sqrt{N} \).

i.e., \( \text{DFT} \ x_m = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} \cdot m \cdot n} \)

IDFT \( \ x_n = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} z(n) e^{j \frac{2\pi}{N} \cdot m \cdot n} \)
IDFT: Apply Parseval’s synthesis identity

\[ x[n] = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} z(m) e^{j \frac{2\pi}{N} mn} \]

⇒ Add up the projections multiplied by the corresponding basis vector.

\[ x[1] = z(1) e^{j \frac{2\pi}{N} 1n} + z(2) e^{j \frac{2\pi}{N} 2n} + z(3) e^{j \frac{2\pi}{N} 3n} \]
Analogy:

- Cook Italian food with 2 different "Basis ingredients".
- DFT helps go from one basis to another.
(3) Visualizing DFT, Bandwidth & Resolution

(3) $Z_m$ is a complex vector

- $|Z_m|$:
  - DC
  - or avg. of signal $x(n)$

- Phase $\Phi Z_m$:
  - Starting point of that rotating stick

Max freq. of a given signal = BANDWIDTH

Analogy for Phase:

$\begin{align*}
\overline{\square} &= \square + \square \\
\overline{\square} &= \square + \Phi
\end{align*}$

Q: Does a time domain signal have phase?
Translating to real-world frequencies

Say sampling freq. = \( f_s \)

Now, slowest freq. = \( N \) samples/cycle

\( N \) samples take \( N \cdot \frac{1}{f_s} \) time

1 cycle takes \( \frac{N}{f_s} \) time

\( \therefore \) Slowest freq. = \( \frac{f_s}{N} \) Hz.
Increasing freq. @ 0, \( \frac{fs}{N} \), \( \frac{2fs}{N} \), \( \frac{3fs}{N} \) ... \( \frac{(N-1)fs}{N} \)

Thus, when analyzing a given signal, we have two knobs.

- **Sampling freq.** \( fs \)
- **No. of FFT points** \( N \)

Important to note:
1. Large \( fs \) means we can see till large freq. component \( fs \)
2. Large \( N \) implies we have better resolution of \( fs \) (i.e., \( fs/N \))
Everyday signal $\cos 2\pi f_m t$

Discrete sampled signal $x[n] = \cos 2\pi f_m n t_s$
where $t_s = \frac{1}{f_s}$ is sampling interval, $f_s$ is sampling freq.

Now what is the DFT ($x[n] = \cos 2\pi f_m n t_s$)?

Real signals always symmetric because the imaginary component needs to be cancelled out by a much faster moving rotation.

Also, phase ZERO because all rotations start from zero.
What is DFT \( x[n] = e^{j2\pi f_1 n} \)?

Just one stick rotating at \( f_1 \) is good enough to create this signal.

Now, what is DFT \( x[n] = \sin(2\pi f_1 n) \)?

Add up to 2\( x \) but multiplied by \( x_m = \frac{1}{2} \)
signals containing full cycles

When we took the DFT of the $\cos(2\pi f_0 n)$ signal, the implicit assumption is that we took $N$ points that exactly covers integer no. of $\cos()$ cycles.

If we take $N$ samples that only covers part of the signal, say

Then, we can no longer form this signal with just the fo freq. component.

Instead, this signal $Y$ can be viewed as a truncated $X$ signal, truncated by a pulse train of $N$ samples:

$\therefore \ DFT(Y) = DFT(X \cdot \text{PulseTrain})$
$\neq DFT(X) \quad \text{So be careful.}$
(3) Properties of DFT

1. DFT is linear:
   \[ \text{DFT} \left( x[n] + y[n] \right) = \text{DFT} (x[n]) + \text{DFT} (y[n]) = X_m + Y_m \]

2. For real signals, DFT is symmetric (negative freq.):
   \[ \text{DFT} (x[n]) = X_m + Y_m \]

Intuitively, the second stick is rotating in the clockwise direction to cancel the imaginary parts of the real signal.

Real signals have symmetric spectrogram because imaginary part cancelled by faster moving stick.
DFT of shifted signal is original DFT with just a phase shift.

\[ x'[n] = x[n+k] \]

i.e., \( x'[n] \) is shifted by \( k \) samples. Then

\[ y[n] = x[n+k] \]

\[ y_m = \sum_{n=0}^{N-1} x[n+k] e^{-j \frac{2\pi}{N} m n} \]

\[ = \sum_{n=0}^{N-1} x[n+k] e^{-j \frac{2\pi}{N} n(m+k)} \]

\[ Y_m = e^{j \frac{2\pi}{N} m.k} x_m \]

means a phase shift

Note: Phase shift can be expressed as \( e^{j m \phi} \) since \( K \) is constant.

\[ Y_1 = e^{j \phi} x_1, \quad Y_2 = e^{j 2\phi} x_2 \quad \ldots \quad Y_m = e^{j m \phi} x_m \]

This means the sticks at different frequencies are all starting at \( k \) steps later, but \( k \) steps means different phases for different freq. say \( k=2 \).

Each stick starting at diff. starting locations.
3) Say signal \( X = [x_0 \ x_1 \ x_2 \ \ldots \ x_{n-1}] \)

\( Y = X_{\text{shifted by 1 index}} = [x_1 \ x_2 \ \ldots \ x_{n-1} \ x_0] \)

3) How to get \( Y \)?

**Note:** 

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
\cdots & \cdots & \cdots & \cdots \\
1 & 1 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
z_0 \\
z_1 \\
z_{n-1}
\end{bmatrix}
= 
\begin{bmatrix}
x_0 \\
x_1 \\
\vdots \\
x_{n-1}
\end{bmatrix}
\]

\[
x_0 = z_0 e^{j\theta_0} + z_1 e^{j\theta_0} + z_2 e^{j\theta_0} + \cdots + z_{n-1} e^{j(n-1)\theta_0}
\]

\[
x_1 = z_0 e^{j2\theta_0} + z_1 e^{j2\theta_0} + z_2 e^{j2\theta_0} + \cdots + z_{n-1} e^{j2(n-1)\theta_0}
\]

\[
x_3 = z_0 e^{j4\theta_0} + z_1 e^{j4\theta_0} + z_2 e^{j4\theta_0} + \cdots + z_{n-1} e^{j4(n-1)\theta_0}
\]

3) Now, let's multiply \( Z_m \) by a phase proportional to \( m \)

\[
\Rightarrow Z_m e^{jm\theta}
\]

\[
y_0 = z_0 e^{j0\theta} e^{j0} + z_1 e^{j\theta} e^{j0} + z_2 e^{j2\theta} e^{j0} + \cdots + z_{n-1} e^{j(n-1)\theta} e^{j(n-1)\theta}
\]

\[
y_1 = z_0 e^{j\theta} e^{j0} + z_1 e^{j2\theta} e^{j\theta} + z_2 e^{j4\theta} e^{j2\theta} + \cdots
\]

\[
\vdots
\]

\[
y_0 = z_0 e^{j(0)\theta} + z_1 e^{j\theta} + z_2 e^{j2\theta} + \cdots + z_{n-1} e^{j(n-1)\theta} = X_1
\]

\[
y_1 = z_0 e^{j\theta} + z_1 e^{j2\theta} + z_2 e^{j4\theta} + \cdots + z_{n-1} e^{j2(n-1)\theta} = X_2
\]

I have shifted \( X \) by 1 index to make \( Y \)

To shift \( K \) indices, multiply by \( Z_m e^{jk\theta K} \).
Now, what happens when you do $X(t)e^{j\phi}$?

Ans: 

$$
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
f_0 & f_1 & f_2 & \cdots & f_{n-1} \\
1 & 1 & 1 & \cdots & 1 \\
\end{bmatrix}
\begin{bmatrix}
Z_0 \\
Z_1 \\
\vdots \\
Z_{n-1} \\
\end{bmatrix}
= 
\begin{bmatrix}
x_0 \\
x_1 \\
\vdots \\
x_{n-1} \\
\end{bmatrix}
\begin{bmatrix}
e^{j\phi} \\
e^{j\phi} \\
\vdots \\
e^{j\phi} \\
\end{bmatrix}
\begin{bmatrix}
e^{j\phi} \\
e^{j\phi} \\
\vdots \\
e^{j\phi} \\
\end{bmatrix}
= 
\begin{bmatrix}
y_0 \\
y_1 \\
\vdots \\
y_{n-1} \\
\end{bmatrix}
$$

This means every stick of increasing freq. getting shifted by same initial phase $\phi$.

From last pg. we know that this will not create the shifted signal. This will create some scrambled signal.

Note: Only when $x$ is a complex phasor (meaning it has only one complex freq, such as $e^{j2\pi ft}$, then multiplying with $e^{j\phi}$ in time domain makes sense).

Even if $X_t$ is a tone (e.g., $\cos 2\pi ft$), it will still have 2 freq. components, hence you need to multiply with $e^{j\phi}$ and $e^{-j\phi}$, respectively, for freq. $f_1$ and $f_{n-1}$.
Meaning of multiplying a signal with $e^{j\phi}$

Note: Any signal $x(t)$ multiplied by $e^{j\phi}$ makes $x(t)$ delayed by $\phi$.

$$x(t) \cdot e^{j\phi} = \left( \sum_f x_f e^{j \frac{2\pi}{N} f \cdot t} \right) e^{j\phi} = \sum_f x_f e^{j \left( \frac{2\pi}{N} f t + \phi \right)}$$

Note: This is a sum of all sticks (each at different freq.)

$e^{j \left( \frac{2\pi}{N} ft + \phi \right)}$ means each stick displaced initially by equal phase $\phi$.

Hence, the result is some scrambled signal, not a shifted version of the original sig, unless $X_f$ is a single freq. phasor.

- Multiplication in time = Conv. in freq.
- vice versa

Not taught yet
Real Spectrogram

Voice Signal.

shows various freq. components at this time instant

shows how a specific freq. comes and goes over time.
Nyquist’s Sampling

Say you are given the following samples and asked to reconstruct the analog signal... and say values match \( \sin(\cdot) \)

Amplitude

\[ t_s \]

\[ \text{time} = \frac{1}{10} \text{ seconds} \]

We can say freq \( f = 10 \text{ Hz} \)

Analog signal \( = \sin(2\pi f n_t s) \)

But is this the only signal that fits these values?

Observe: \( \sin(2\pi f n_t s + 2\pi m) \) also fits the values given, if \( m = \text{integer} \)

i.e., \( \sin(2\pi (f_0 + \frac{m}{n_t}) n_t s) \) fits values

Choosing \( m \) as multiple of \( n \) \( (\frac{m}{n} = k) \)

\[ \sin(2\pi (f_0 + k \cdot \frac{1}{t_s}) n_t s) \]

\[ = \sin(2\pi (f_0 + k f_s) n_t s) \ldots f_s \text{ i.e. sampling freq.} \]
(3) This means, many other frequencies, separated by $kfs$ from the core freq. $f$ fits the given set of points.

\[ \downarrow \]

Called "ALIASED" frequencies.

ie., \((f + kfs)\) are aliases of \(f\)
for different values of \(k\).

\[ \downarrow \]

If I sample a signal at $fs$, and therefore get to see freq. components \(\frac{fs}{N}, \frac{2fs}{N}, \frac{3fs}{N}, \ldots, \frac{(N-1)fs}{N}\),
actually, each of these freq. components could have aliased freq. from other freq. contained in the signal.

(3) To ensure we remove aliasing (which pollutes) we need to understand max freq. component of signal, say $fw$. 

\[ -fw \quad fw \]
Observe that higher the $fs$, more separated are the aliased freq. Example

$$f - 2fs \ f - fs \ f \ f + fs \ f + 2fs \ f + 3fs \ldots$$

Now, to avoid pollution for a given signal of bandwidth $fw$, we need no. freq. within it to alias to within its bandwidth.

That is $fw$ should alias to left of $-fw$ and $-fw$ should alias to right of $fw$

This implies: $fw - fs < -fw$

or $-fw + fs > fw$

\[ \downarrow \]

$fs > 2fw$

i.e., sampling freq. must be twice b/w to have an unpolluted reconstruction of the signal.
Thus, sampling at $f_s$ may show the alias of $(-fw+f_s)$.

However, knowing $fw$, we can filter out the signal at $fw$, thereby removing all the signals outside $[-fw,fw]$ giving us a full reconstruction of the signal of interest.

\[ \text{Low Pass filter (LPF)} \]

\[ \text{alias free original signal} \]
Even when $f_s = 2f_w$, some $f_w + \Delta$ can still alias to $-f_w + \Delta$. What then?

Ans: Use an anti-aliasing filter first to remove all signals outside the max freq. component $f_w$.

This is an analog LPF.

Then, for the $f_w$ bandwidth signal, sample this at $2xf_w$ so that none of it’s own frequency components alias into this band.

Net result: Outside freq. removed by LPF and internal freq. removed by Nyquist sampling.
The signal processing flow:

1. **Input Signal** (e.g., voice) $X(t)$
2. Filter at $f_w$ ($\approx 20\text{kHz}$)
3. Nyquist Sample
4. Sample at $f_s \geq 40\text{kHz}$
5. Digital Samples $x[n]$
6. $N$ point FFT, say $N = 1024$
7. Visualize $X(\omega)$ and $|X(\omega)|$
Questions?
Quick examples:
- \[ \cos 2\pi f_s t + \sin 2\pi f_m t + \cos(2\pi f_n t + \phi) \]
- Tell me \( x \) given a DFT
- Rectangular pulse signal, impulse train
- Converting \( m \) to actual freq. \( m \frac{f_s}{N} \)
- Significance of \( f_s \) and \( N \)
- View angle and resolution
- Nyquist

Putting it all together ⇒ show example.
- Signal, Nyquist sample, LPF, N point FFT, \( |X_m|, fX_m \), Spectrogram

Practical scenarios
- N points not at boundaries of \( \sin \) and \( \cos \)
- Can be viewed as multiplication

Convolution
- Impulse response

Properties of convolution (duality, freq. shift)

Wireless channel (LOS) ⇒ \( P_r = P_t / n_e \)

Multipath
- Estimating \( H \), noise

\( \text{UR, CFR,} \)
- Estimate \( H_f \) only where energy is present in \( X_f \)
Topics

1. Convolution
2. The lens of convolution
   - Averaging
   - Impulse response
3. Properties of convolution (time, freq.)
4. Convolution examples and quiz.
   - \( \cos(2\pi f t) * \cos(2\pi f mt) \) ?
   - \( w(t) * \cos(2\pi f t) \) ?
   - \( w(t) * \cos(2\pi f t) \) ?
   - \( \cos(2\pi f t) * \cos(2\pi f mt) \) ?
5. Wireless channel
   - Pathloss index
   - Line of sight
   - Multipath channel
6. CIR and CFR
7. Estimating channel \( H_f \)
   - Without noise
   - With noise
   - Estimate \( H_f \) only for \( X_f \) freq.
Consider a signal averaging operation

\[ y_0 = \text{n/a} \]
\[ y_1 = \text{something} \]
\[ y_2 = \frac{x_0 + x_1 + x_2}{3} \]
\[ y_3 = \frac{x_1 + x_2 + x_3}{3} \]

Think of this as

\[ y(t) = \sum_{\tau=0}^{2} h(\tau) x(t-\tau) \Rightarrow y(\bullet) = h \ast x \]

i.e., the \textit{convolution} impulse response of the system.
Convolution Steps:
1. Take a signal and flip it by x-axis.
2. Multiply and add all the values.
3. Shift the signal step by step, and repeat multiply & add.

Convolution:
\[ y_n = \sum_{i=0}^{w} x(n-i) h(i) \]
Multiplication & Conv.

\[ \text{FFT}(y_n = h_n \ast x_n) = H(m) \cdot X(m) \]

\[ \text{IFFT}(Y(m) = H(m) \odot X(m)) = h_n \ast x_n \]

\[ \text{FFT}(h_n \ast x_n) = H(m) \ast X(m) \]

\[ \text{IFFT}(H(m) \ast X(m)) = h_n \cdot x_n \]

Example: \[ \text{FFT}(\cos 2\pi f_c t) = \]

Say a signal \[ m(t) = \]

Say \[ z(t) = m(t) \cdot \cos 2\pi f_c t. \] (time domain multiply)

Remember, a signal convolved with an impulse signal, gives you the original signal.
Convolution in freq. domain

Mf

Yf

\[ M_f = Y_f \]

\[ Y_0 = 0 \]
\[ Y_1 = 0 \]
\[ Y_2 = 1.1 \]
\[ Y_3 = 1.2 \]
\[ Y_4 = 1.3 \]
\[ Y_5 = 1.2 \]
\[ Y_6 = 1.1 \]
\[ Y_7 = 0 \]

Similarly

\[ Y_{-1} = 0 \]
\[ Y_{-3} = 1.2 \]
\[ Y_{-2} = 1.1 \]
\[ Y_{-4} = 1.3 \]...
Wireless Channel:

\[ P_R = \frac{P_T}{r^n} \]

\( r^n \) = pathloss index
\( 2 \leq n < 4 \)

Why \( r^n \)? Because energy dissipates.
Area of Sphere: \( 4\pi r^2 \)
More realistic: Multipath channel

$$y[n] = \sum_{i=0}^{N-1} h(i) x(n-i)$$

Even more realistic:

$$y[n] = h[n] \ast x[n] + \text{noise}$$

hardware noise (Random process)
So what does the channel look like in time domain?

\[ \text{IFFT}(H_f) = h[n] \]

called **CIR**: Channel Impulse Response.

\[ H_f : \text{Called CFR : Channel Freq. Response} \]

This is like saying, if I transmitted a unit pulse from Tx, what would I receive at Rx.

Ans: The CIR.

Q: Can \( h[5] \) be greater than \( h[j], j < 5 \)? Can \( h[5] \) be greater than \( h[0] \)?

Ans: Yes for both since attenuation also a \( f_t \) of reflecting material, & \( h[0] \) may be passing through a wall.
This means, channel also a signal $h[n]$.

Rx receives $y[n]$ which is a convolution of the transmitted and channel signal.

Now recall $\text{FFT}(x_n * h_n) = X_f H_f$

$Y_f = H_f X_f$ for multipath channel.

with noise.

$Y_f = H_f X_f + N_f$

or $Y_f - N_f = H_f X_f = X_f H_f$

$X_f H_f = Y_f - N_f$

known preamble unknown channel measured value

$A \vec{x} = \vec{b}$ but $\vec{b}$ not in col. space of $A$

Apply projection/regression and solve for $H_f$. 
Of course, note that $X_f H_f$ is not a dot product, instead it's an element by element multiplication that gives you a vector.

i.e.,

\[
X_{f_1} \cdot H_{f_1} = Y_{f_1}
\]
\[
X_{f_2} \cdot H_{f_2} = Y_{f_2}
\]
\[
X_{f_3} \cdot H_{f_3} = Y_{f_3}
\]

\[\vdots\]

\[
X_{f_1} = \frac{Y_{f_1}}{H_{f_1}} = \frac{e^{j\theta}}{e^{j\varphi}} = e^{j(\theta-\varphi)}
\]

To write this as a dot product, we can do

\[
X_f = \begin{bmatrix} X_{f_1} & 0 & 0 & \cdots & 0 \\ 0 & X_{f_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & X_{f_n} \end{bmatrix}
\]

Then,

\[
X_f \cdot H_f = Y_f
\]
You see only where you shine light.

\[ Y_f = H_f x_f + N \implies \hat{H}_f = \frac{Y_f}{x_f} \]

This \( x_f \) is like shining light in a freq. band covered by the \( x_f \) signal.

Estimated \( \hat{H}_f \) is also only in that band.

You do not get to see the channel \( H_f \) in other freq. bands on which you have not shone any light.
What happens when channel is changing?

\[ y_t = h_0 x_t + h_1 x_{t-1} + h_2 x_{t-2} \ldots \]
\[ y_{t+1} = h_0 x_{t+1} + h_1 x_t + h_2 x_{t-1} \ldots \]
\[ y_{t+2} = h_0 x_{t+2} + h_1 x_{t+1} + h_2 x_t \ldots \]

If you know \( x_t \) then you estimate the channel at every time instant.

For blind estimation, you guess

\[ [h_0 \ h_1 \ h_2 \ldots \ h_n] \]

and then \ldots (??)

Difference

\[ \text{fading} \ vs. \ \text{fast fading} \]

en reflected signal is very near los vs. very far, what does \( H \) look like?

Combination of different echoes of a signal cannot produce a freq. not present in the signal.