Fourier Transform

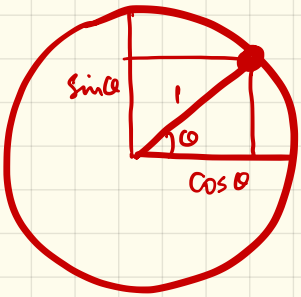
⊙ Abstractions, complex no., Rotation, $e^{j\theta}$

⊙ Model multiplicity → count / numbers.

⊙ Model opposite → I have \$5
How much should I add to make \$0? Ans: −5

Numbers & neg. numbers are all abstractions of reality

Now, How to model rotation?



We have $\cos\theta$ and $\sin\theta$.
Of course we need to express their relation in some way. We cannot just say

$\cos\theta + \sin\theta$ ✗

or $\langle \cos\theta, \sin\theta \rangle$

Which still does not allow algebra.

OK, we can say $\vec{x}\cos\theta + \vec{y}\sin\theta$

But we don't have to say $\vec{x}$, we just need to make $\vec{y}$ become $\perp$ to $\vec{x}$.

How? Well, one way is to say that $\vec{y}\cdot\vec{y}$ should be $-1.\vec{x}$.

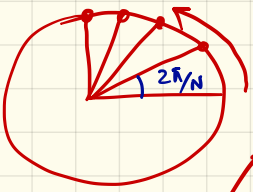
i.e., $y^2 = -1$ $\therefore y = \sqrt{-1} = j$

$\therefore$ Rotation is a vector of $\cos\theta + j\sin\theta$

Aha, this is exactly $= e^{j\theta}$ ⇒ models rotation

Fourier Matrix : An orthogonal Basis

Now, what would rotation at different freq. look like?

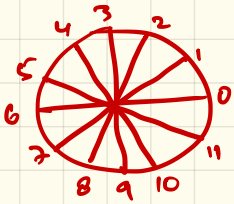


Say we want to model every freq. in N discrete steps.

One cycle/s

$$
\begin{array}{c}
 \\ n=0 \\ n=1 \\ n=2 \\ \vdots \\ n=N-1
\end{array}
\underbrace{
\begin{array}{c}
\text{freq } 0 \\
\begin{bmatrix} e^{j0} \\ e^{j0} \\ e^{j0} \\ \vdots \\ e^{j0} \end{bmatrix}
\end{array}
\begin{array}{c}
\text{freq. } f_1 \\
\begin{bmatrix} e^{j0\cdot\frac{2\pi}{N}} \\ e^{j1\cdot 2\pi/N} \\ e^{j2\cdot 2\pi/N} \\ \vdots \\ e^{j(N-1)\cdot\frac{2\pi}{N}} \end{bmatrix}
\end{array}
\begin{array}{c}
\text{freq } 2f_1 \\
\begin{bmatrix} e^{j0\cdot\left(2\left(\frac{2\pi}{N}\right)\right)} \\ e^{j1\cdot\left(2\cdot\frac{2\pi}{N}\right)} \\ e^{j2\left(2\cdot\frac{2\pi}{N}\right)} \\ \vdots \\ e^{j(N-1)\left(2\cdot\frac{2\pi}{N}\right)} \end{bmatrix}
\end{array}
\cdots
\begin{array}{c}
\text{freq } mf_1 \\
\begin{bmatrix} e^{j0\left(m\frac{2\pi}{N}\right)} \\ e^{j\cdot 1\cdot\left(m\frac{2\pi}{N}\right)} \\ e^{j\cdot 2\cdot\left(m\frac{2\pi}{N}\right)} \\ \vdots \\ e^{j(N-1)\left(m\frac{2\pi}{N}\right)} \end{bmatrix}
\end{array}
\begin{array}{c}
\text{freq } (N-1)f \\
\vdots
\end{array}
}_{\text{Fourier observed each column is orthogonal.}}
$$

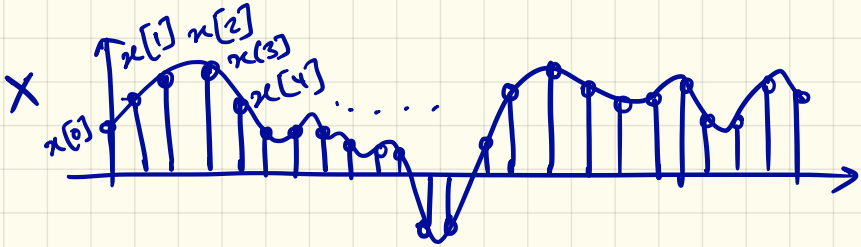
$f_0 \cdot f_1 = 0 \Rightarrow$ you can immediately see because



0	cancels	6
1	cancels	7
2	"	8
3	"	9
4	"	10
5	"	11

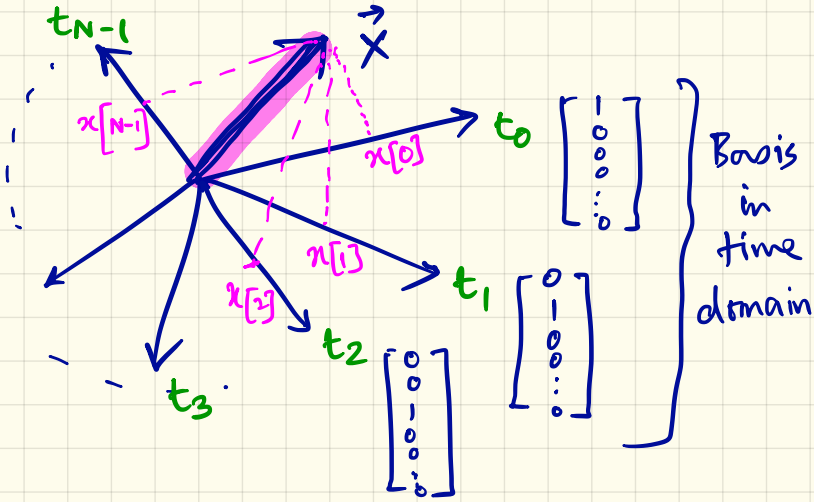
Same holds for any pair.

⊖ Vector Projection to Fourier Basis



$$\vec{X} = \left[x[0], x[1], x[2] \; \ldots \; x[N-1] \right]$$

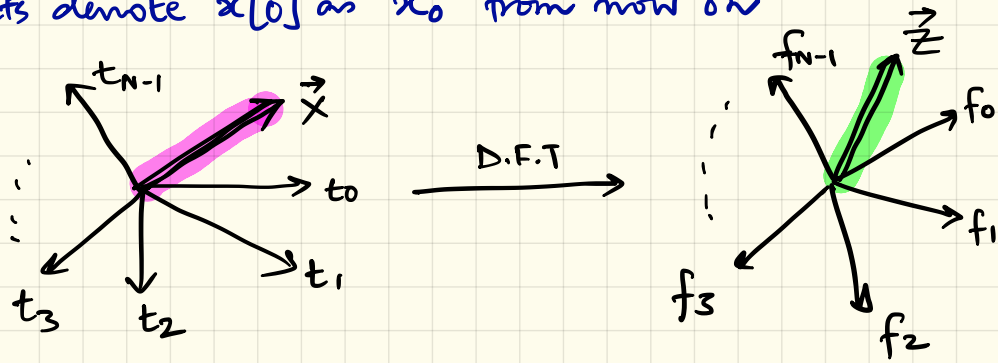
Vector in N dimensional space.



$$\therefore \quad \vec{X} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ 0 & 0 & 0 & & \vdots \\ \vdots & \vdots & \vdots & & 1 \\ 0 & 0 & 0 & & \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

(→) Let's denote $x[0]$ as x_0 from now on

(→)



$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ 0 & 0 & 0 & & \vdots \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{bmatrix} = \underbrace{\begin{bmatrix} | & | & | & & | \\ f_0 & f_1 & f_2 & \cdots & f_{N-1} \\ | & | & | & & | \end{bmatrix}}_{\text{Orthogonal Fourier Basis matrix}} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \\ z_{N-1} \end{bmatrix}$$

$$\therefore \quad z(m) = F^{-1} \cdot \vec{x}$$

$$z(m) = (F^*)^T \cdot \vec{x}$$

$$\therefore \quad \underbrace{\begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \\ z_{N-1} \end{bmatrix} = \begin{bmatrix} \text{———} & f_0^* & \text{———} \\ \text{———} & f_1^* & \text{———} \\ \text{———} & f_2^* & \text{———} \\ \vdots & \vdots & \\ \text{———} & f_{N-1}^* & \text{———} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{bmatrix}}_{\text{Discrete Fourier Transform (DFT)}}$$

DFT Equations

Consider 3D case :

$$\begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} — & f_0^* & — \\ — & f_1^* & — \\ — & f_2^* & — \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}$$

$$z_0 = e^{-j\frac{2\pi}{N}.0.0}.x_0 + e^{-j\frac{2\pi}{N}.0.1}.x_1 + e^{-j\frac{2\pi}{N}.0.2}x_2$$

$$z_1 = e^{-j\frac{2\pi}{N}.1.0}.x_0 + e^{-j\frac{2\pi}{N}.1.1}x_1 + e^{-j\frac{2\pi}{N}.1.2}.x_2$$

$$z_2 = \underbrace{e^{-j\frac{2\pi}{N}.2.0}x_0 + e^{-j\frac{2\pi}{N}.2.1}.x_1 + e^{-j\frac{2\pi}{N}.2.2}.x_2}$$

DFT :

$$Z(m) = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}.m.n}$$

IDFT :

$$x[n] = \sum_{m=0}^{N-1} Z(m)\, e^{j\frac{2\pi}{N}.m.n}$$

Scaling by $\frac{1}{\sqrt{N}}$

Note that dot.product = Projection only when the dot product is normalized.

Now, for DFT, observe that each column of F is not a unit vector, e.g., $[1\ 1\ 1]^T$.

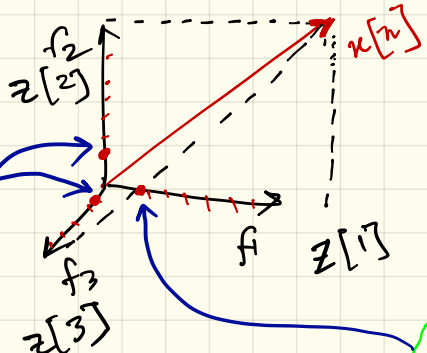
$\therefore$ The DFT needs to be scaled by the length of the vector $1/\sqrt{N}$

i.e., DFT $X_m = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}mn}$

IDFT $x_n = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} X_m\, e^{j\frac{2\pi}{N}mn}$

⊛ IDFT : Apply Parseval's synthesis Identity

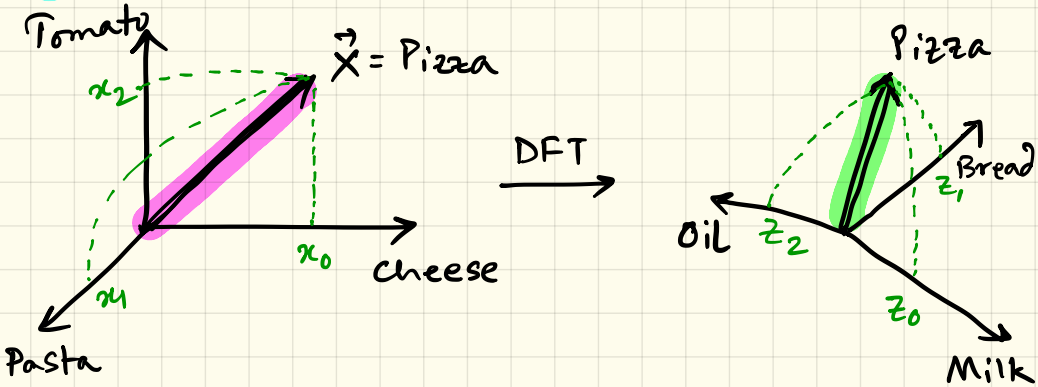
$$x[n] = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} Z(m) e^{j\frac{2\pi}{N} \cdot mn}$$



⇒ Add up the projections multiplied by the corresponding basis vector.

$$x[1] = Z(1) e^{j\frac{2\pi}{N} \cdot 1 \cdot n} + Z(2) e^{j\frac{2\pi}{N} \cdot 2n} + Z(3) e^{j\frac{2\pi}{N} \cdot 3n}$$

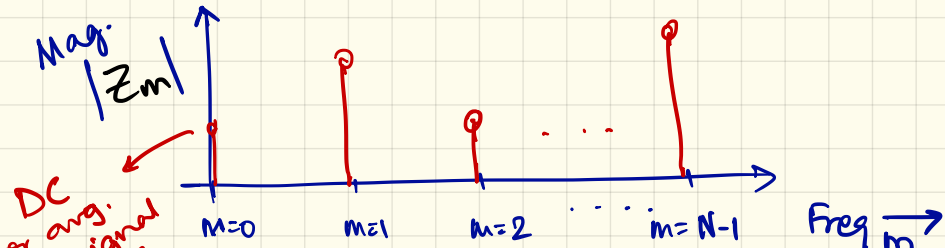
③ Analogy:



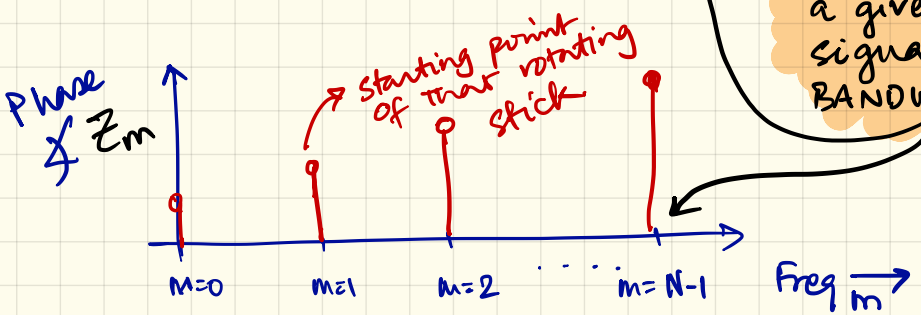
- Cook Italian food with 2 different "Basis ingredients"
- DFT helps go from one basis to another.

Visualizing DFT, Bandwidth & Resolution

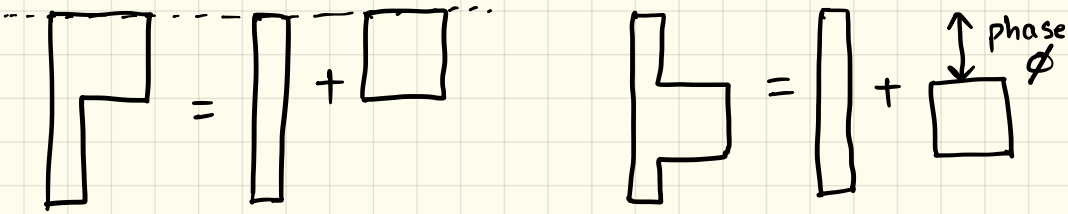
- Z_m is a complex vector



Max freq. of a given signal = BANDWIDTH

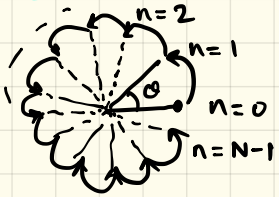


- Analogy for phase :



Q: Does a time domain signal have phase?

⊖ Translating to real-world frequencies



Say sampling freq. = f_s

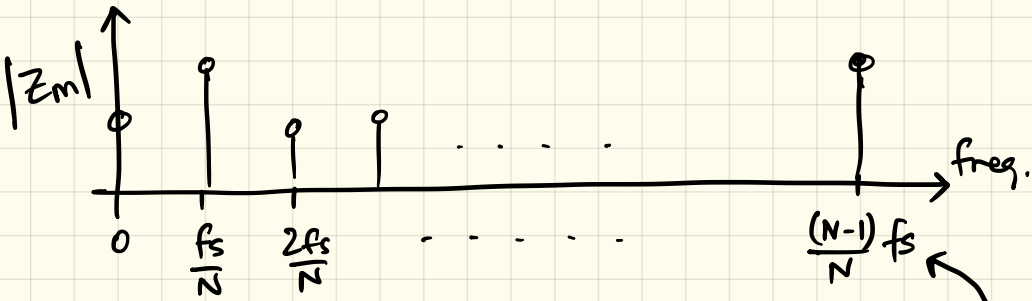
Now, slowest freq = N samples/cycle

N samples take $N \cdot \frac{1}{f_s}$ time

1 cycle takes N/f_s time

$\therefore$ slowest freq. $= \frac{f_s}{N}$ Hz.

→ ∴ Increasing freq. @ $0, \frac{f_s}{N}, \frac{2f_s}{N}, \frac{3f_s}{N} \ldots \frac{(N-1)f_s}{N}$



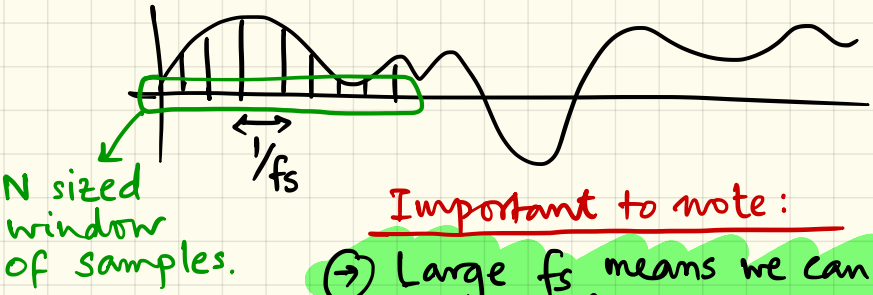
$\Downarrow$

$\approx f_s$ for large N

→ Thus, when analyzing a given signal, we have two knobs.

→ Sampling freq. f_s

→ No. of FFT points N



Important to note:

→ Large f_s means we can see ti'll large freq. components f_s

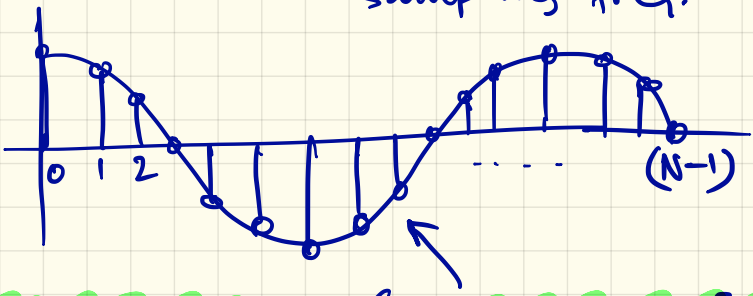
→ Large N implies we have better resolution of f_s (i.e., f_s/N)

3 Examples of DFT : Cos, $e^{j\theta}$, Sin

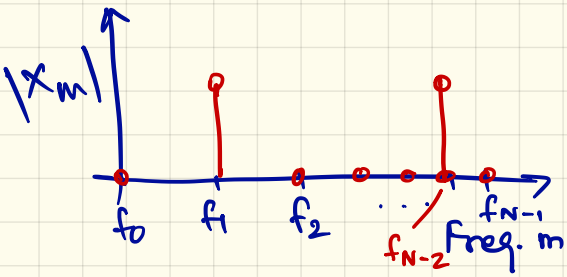
Everyday signal $\cos 2\pi f_m t$

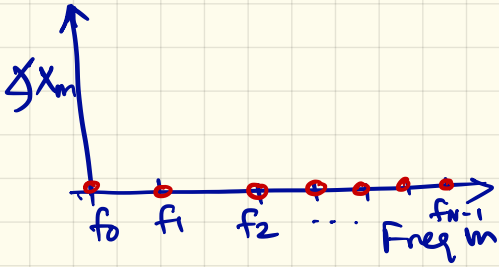
Discrete sampled signal $x[n] = \cos 2\pi f_m n t_s$

where $t_s = \frac{1}{f_s}$ is sampling interval, f_s is sampling freq.

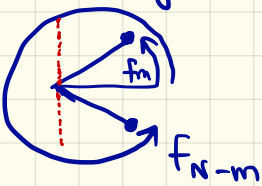


Now what is the DFT ($x[n] = \cos 2\pi f_1 n t_s$)



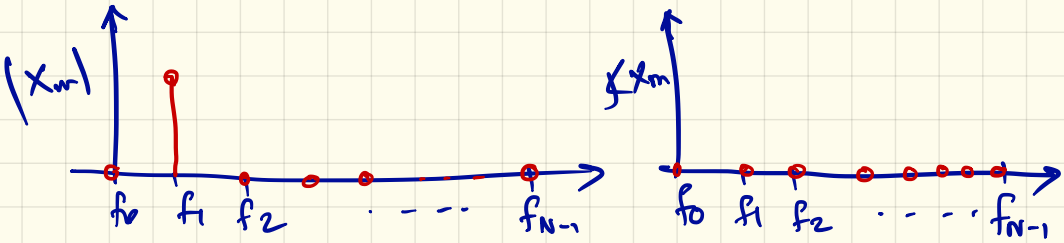


Real signals always symmetric because the imaginary component needs to be cancelled out by a much faster moving rotation



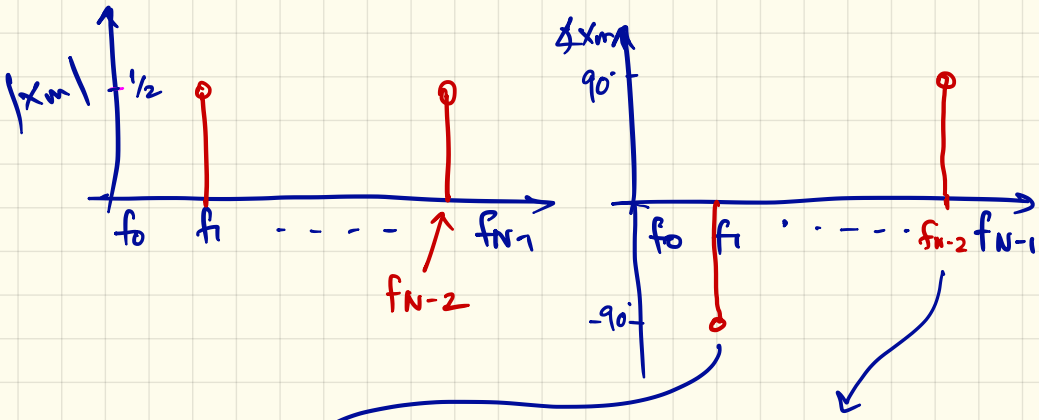
Also, phase ZERO because all rotations start from zero.

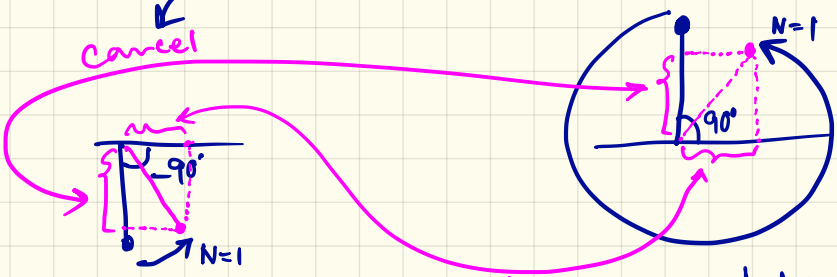
(→) What is DFT($x[n] = e^{j2\pi f_1 n t_s}$) ?



Just one stick rotating at f_1 is good enough to create this signal.

(→) Now, what is DFT($x[n] = \sin 2\pi f_1 n t_s$) ?



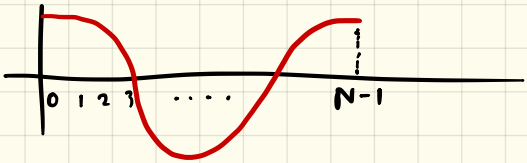


add up to 2x but multiplied by $x_m = 1/2$

⊖ Signals containing full cycles

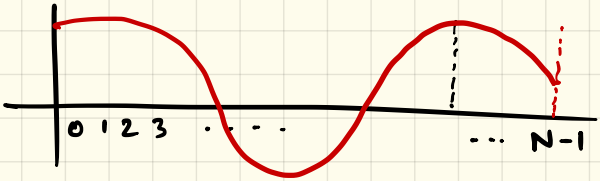
When we took the DFT of the $\cos(2\pi f_0 n t_s)$ signal, the implicit assumption is that we took N points that exactly covers integer no. of $\cos(\cdot)$ cycles.

$x =$



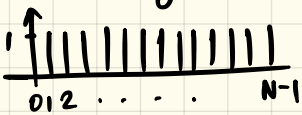
⊖ If we take N samples that only covers part of the signal, say

$y =$



Then, we can no longer form this signal with just the f_0 freq. component.

⊖ Instead, this signal y can be viewed as a truncated x signal, truncated by a pulse train of N samples



$$\therefore \quad \mathrm{DFT}(y) = \mathrm{DFT}(x \cdot \text{PulseTrain})$$
$$\neq \mathrm{DFT}(x)$$

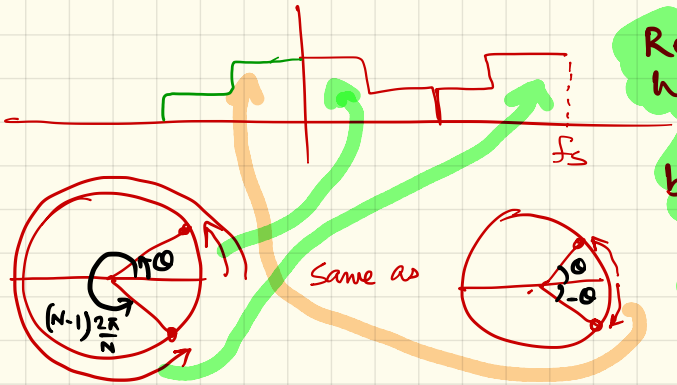
... So be careful.

⊛ Properties of DFT

① DFT is linear :

$$DFT(x[n] + y[n]) = DFT(x[n]) + DFT(y[n])$$
$$= X_m + Y_m$$

② For real signals, DFT symmetric (negative freq.)

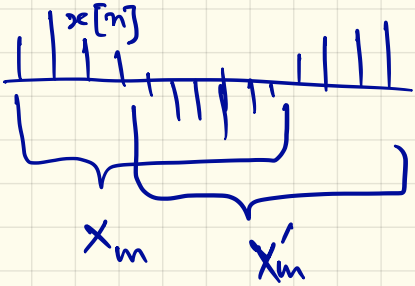


Real signals have symm. spectrogram because imaginary part cancelled by faster moving stick.

So this is equivalent to taking the mirror image and pasting it on the (-)ve freq. axis.

Intuitively, the second stick is rotating in the clockwise direction to cancel the imaginary parts of the real signal.

③ DFT of shifted signal is original DFT with just a phase shift.



$$x'[n] = x[n+k]$$

i.e., $x'[n]$ is shifted by k samples. Then

$$y[n] = x[n+k]$$

$$\therefore \quad Y_m = \sum_{n=0}^{N-1} x[n+k]\, e^{-j\frac{2\pi}{N}.m.n}$$

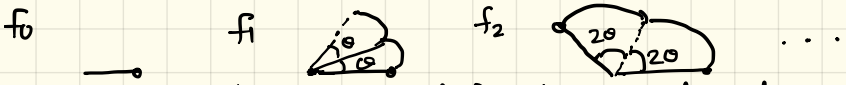
$$= \sum_{n=0}^{N-1} x[n+k]\, e^{-j\frac{2\pi}{N}.m.(n+k)} \cdot e^{j\frac{2\pi}{N}.mk}$$

$$\boxed{Y_m = \underbrace{e^{j\frac{2\pi}{N}.m.k}}_{\text{means a phase shift}} X_m}$$

⊖ **Note:** Phase shift can be expressed as $e^{jm\phi}$ since k is constant.

$\therefore \quad Y_1 = e^{j\phi} X_1 \quad , Y_2 = e^{j2\phi} . X_2 \quad \ldots \quad Y_m = e^{jm\phi} . X_m$

This means the sticks at diff. frequency are all starting at k steps later, but k steps means different phases for different freq. Say $k=2$.



Each stick starting at diff starting locations.

Zoom out for clarity

- Say signal $X = [x_0 \quad x_1 \quad x_2 \quad \dots \quad x_{n-1}]$

$Y = X_{\text{shifted By 1 index}} = [x_1 \quad x_2 \quad \dots \quad x_{n-1} \quad x_0]$

- How to get Y ?

Note :

$$\begin{bmatrix} | & | & & | \\ f_0 & f_1 & \cdots & f_{n-1} \\ | & | & & | \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

$$x_0 = z_0 e^{j0} + z_1 e^{j0} + z_2 e^{j0} + \dots z_{n-1} e^{j0}$$
$$x_1 = z_0 e^{j0} + z_1 e^{j\theta} + z_2 e^{j2\theta} + \dots z_{n-1} e^{j2(n-1)\theta}$$
$$x_3 = z_0 e^{j0} + z_1 e^{j2\theta} + z_2 e^{j4\theta} + \dots z_{n-1} e^{j3(n-1)\theta}$$

- Now, let's multiply z_m by a phase proportional to m $\Rightarrow z_m \cdot e^{jm\theta}$

$$Y_0 = z_0 e^{j0\theta} \cdot e^{j0} + z_1 e^{j\theta} \cdot e^{j0} + z_2 e^{j2\theta} \cdot e^{j0} + \dots z_{n-1} e^{j(n-1)\theta} \cdot e^{j(n-1)0}$$
$$Y_1 = z_0 e^{j0\theta} \cdot e^{j0} + z_1 e^{j\theta} \cdot e^{j\theta} + z_2 e^{j2\theta} \cdot e^{j2\theta} + \dots$$
$$\vdots$$

$$\therefore Y_0 = z_0 e^{j0} + z_1 e^{j\theta} + z_2 e^{j2\theta} \dots e^{j(n-1)\theta} = x_1$$
$$Y_1 = z_0 e^{j0} + z_1 e^{j2\theta} + z_2 e^{j4\theta} \dots e^{j2(n-1)\theta} = x_2$$
$$\vdots$$

I have shifted X by 1 index to make Y

To shift K indices, multiply by $z_m \cdot e^{jmk\theta}$.

(→) Now, what happens when you do $x(t)e^{j\phi}$?

Ans:

$$\begin{bmatrix} | & | & & | \\ f_0 & f_1 & \cdots & f_{n-1} \\ | & | & & | \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

$$\begin{bmatrix} | & | & & | \\ f_0 & f_1 & \cdots & f_{n-1} \\ | & | & & | \end{bmatrix} \underbrace{\begin{bmatrix} z_0 \cdot e^{j\phi} \\ z_1 \cdot e^{j\phi} \\ \vdots \\ z_{n-1} e^{j\phi} \end{bmatrix}} = \begin{bmatrix} x_0 \cdot e^{j\phi} \\ x_1 \cdot e^{j\phi} \\ \vdots \\ x_{n-1} e^{j\phi} \end{bmatrix} = \begin{bmatrix} Y_0 \\ Y_1 \\ \vdots \\ Y_{n-1} \end{bmatrix}$$

This means every stick of increasing freq. getting shifted by same initial phase ϕ.

From last pg. we know that this will not create the shifted signal. This will create some scrambled signal.

<u>Note</u>: Only when x is a complex phasor (meaning it has only one complex freq, such as $e^{j2\pi ft}$, then multiplying with $e^{j\phi}$ in time domain makes sense).

Even if x_t is a tone (e.g., $\cos 2\pi f_1 t$), it will still have 2 freq. components, hence you need to multiply with $e^{j\phi}$ and $e^{-j\phi}$, respectively, for freq. f_1 and f_{n-1}.

Meaning of multiplying a signal with $e^{j\phi}$

Note: Any signal $x(t)$ multiplied by $e^{j\phi}$ makes $x(t)$ delayed by ϕ.

$$x(t).e^{j\phi} = \left(\sum_f X_f e^{j\frac{2\pi}{N}.f.t}\right)e^{j\phi} = \sum_f X_f e^{j\left(\frac{2\pi}{N}ft+\phi\right)}$$

Note: This is a sum of all sticks (each at different freq.)

& $e^{j\left(\frac{2\pi}{N}ft+\phi\right)}$ means each stick displaced initially by equal phase ϕ.

Hence, the result is some scrambled signal, not a shifted version of the original sig, unless X_f is a single freq. phasor.

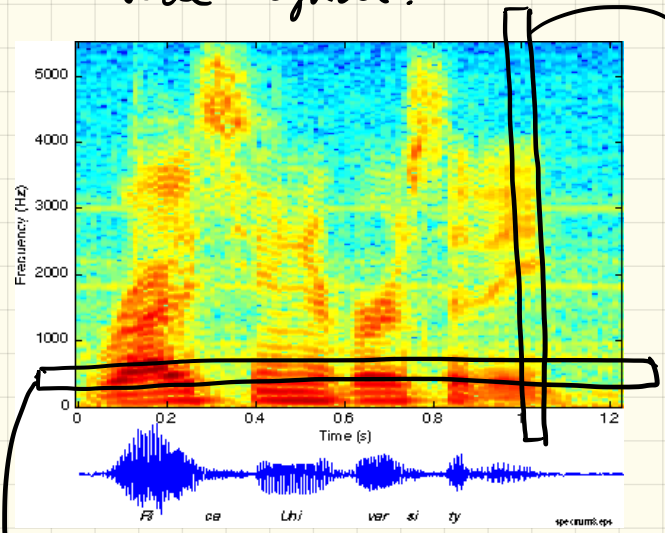
---

(4) Multiplication in time = Conv. in freq. & vice versa

not taught yet

Real spectrogram

Voice signal.

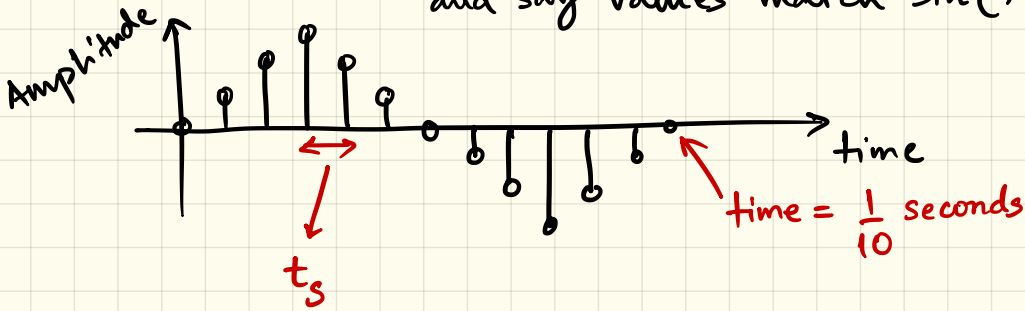


shows various freq. components at this time instant

shows how a specific freq. comes and goes over time.

Nyquist's Sampling

- Say you are given the following discrete samples and asked to reconstruct the analog signal... and say values match $\sin(\cdot)$



- We can say freq $f = 10$ Hz

$\therefore$ Analog signal $= \sin(2\pi f n t_s)$

- But is this the only signal that fits these values?

Observe: $\sin(2\pi f n t_s + 2\pi m)$ also fits the values given, if m = integer

i.e., $\sin\left(2\pi\left(f_0 + \frac{m}{n t_s}\right) n t_s\right)$ fits values

Choosing m as multiple of n $\left(\frac{m}{n} = k\right)$

$$\sin\left(2\pi\left(f_0 + k \cdot \frac{1}{t_s}\right) n t_s\right)$$

$$= \sin\left(2\pi\left(f_0 + k f_s\right) n t_s\right)$$ ... f_s is sampling freq.

→ This means, many other frequencies, separated by kf_s from the core freq. f fits the given set of points.

$\Downarrow$

called "ALIASED" frequencies.

ie., $(f + kf_s)$ are aliases of f for different values of k.

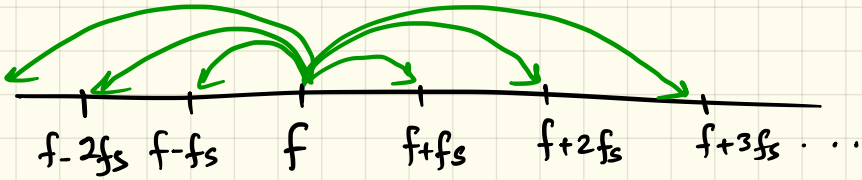
$\Downarrow$

If I sample a signal at f_s, and therefore get to see freq. components of $\frac{f_s}{N}, \frac{2f_s}{N}, \frac{3f_s}{N} \ldots \frac{N-1}{N} f_s$,

actually, each of these freq. components could have aliased freq. from other freq contained in the signal.

→ To ensure we remove aliasing (which pollutes) we need to understand max freq. component of signal, say f_w.

- Observe that higher the f_s, more separated are the aliased freq. Example



- Now, to avoid pollution for a given signal of bandwidth f_w, we need no. freq. within it to alias to within its bandwidth.

- That is f_w should alias to left of $-f_w$ and $-f_w$ should alias to right of f_w

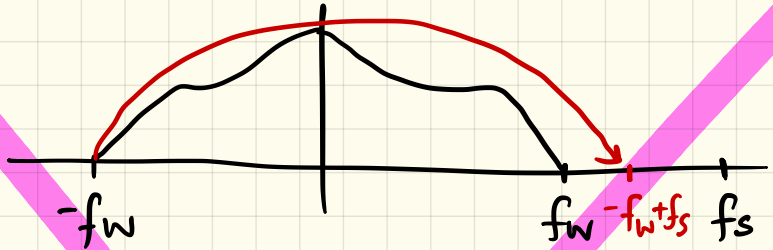
- This implies : $f_w - f_s < -f_w$

or $-f_w + f_s > f_w$

$$\Downarrow$$

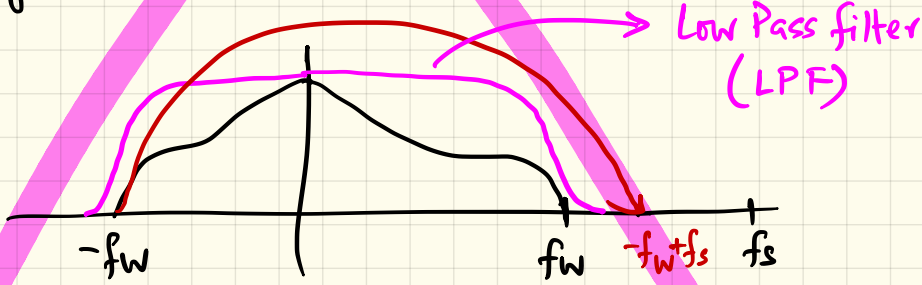
$$\boxed{f_s > 2f_w}$$

i.e., sampling freq. must be twice b/w to have an unpolluted reconstruction of the signal.



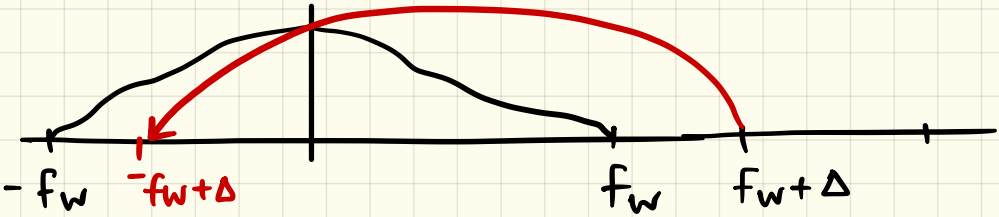
Thus, sampling at f_s may show the alias of $(-f_w+f_s)$.

However, knowing f_w, we can filter out the signal at f_w, thereby removing all the signals outside $[-f_w, f_w]$ giving us a full alias free reconstruction of the signal of interest.



$\Downarrow$

⊕ Even when $f_s = 2f_w$, some $f_w + \Delta$ can still alias to $-f_w + \Delta$. What then?



Ans: Use an anti-aliasing filter first to remove all signals outside the max freq. component f_w.

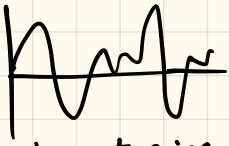
This is an analog LPF

Then, for the f_w bandwidth signal, sample this at $2 \times f_w$ so that none of it's own frequency components alias into this band.

$\Downarrow$

Net result: Outside freq. removed by LPF and internal freq. removed by Nyquist sampling.

⊖ The signal processing flow.



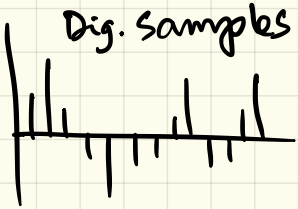
input signal
(e.g., voice)

x_t → **LPF** → **Nyquist sample** →

Filter at f_w (≈ 20 KHz)

Sample at f_s ≥ 40 KHz

$x[n]$

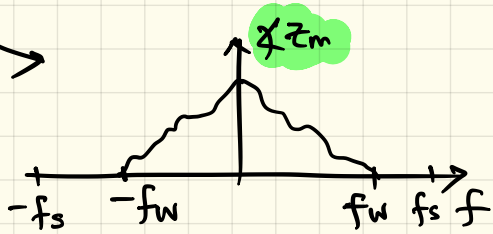


Dig. samples

→ **FFT** →

N point FFT, say $N = 1024$

Z_m → **Visualize** →



Questions?

- Quick examples:
  - $\cos 2\pi f_m t + \sin 2\pi f_m t + \cos(2\pi f_m t + \phi)$
  - Tell me x_t given a DFT
  - Rectangular ~~sig~~ pulse signal. , impulse train
- converting m to actual freq. $\Rightarrow m \frac{f_s}{N}$
- Significance of f_s and N
  - view angle and resolution
- Nyquist
- Putting it all together $\Rightarrow$ show example.
  - signal, Nyquist sample, LPF, N point FFT, $|X_m|$, $\measuredangle X_m$, Spectrogram

- Practical scenarios
  - N points not at boundaries of sin and cos.
  - Can be viewed as multiplication

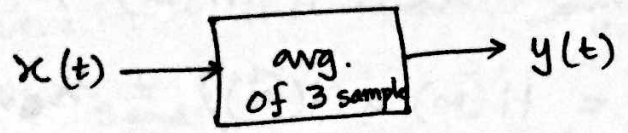
- Convolution
- Impulse response
- Properties of convolution (duality, freq. shift)
- Wireless channel (LOS) $\Rightarrow P_r = P_T / r^{\alpha}$
- Multipath
- Estimating H, noise
- CIR, CFR,
- Estimate H_f only where energy is present in X_f.

Topics

- Convolution
- The lens of convolution
  - Averaging
  - Impulse response
- Properties of convolution (time, freq)
- Convolution examples and quiz.
  - $\cos 2\pi f_c t \; * \; \cos 2\pi f_m t$?
  - $m(t) \; * \; \cos 2\pi f_c t$?
  - $m(t) \cdot \cos 2\pi f_c t$?
  - $\cos 2\pi f_c t \cdot \cos 2\pi f_m t$?
- Wireless channel
  - Pathloss index
  - Line of sight
  - Multipath channel
- CIR and CFR
- Estimating channel H_f
  - Without noise
  - With noise.
  - Estimate H_f only for X_f freq.

CONVOLUTION

⊛ Consider a signal averaging operation

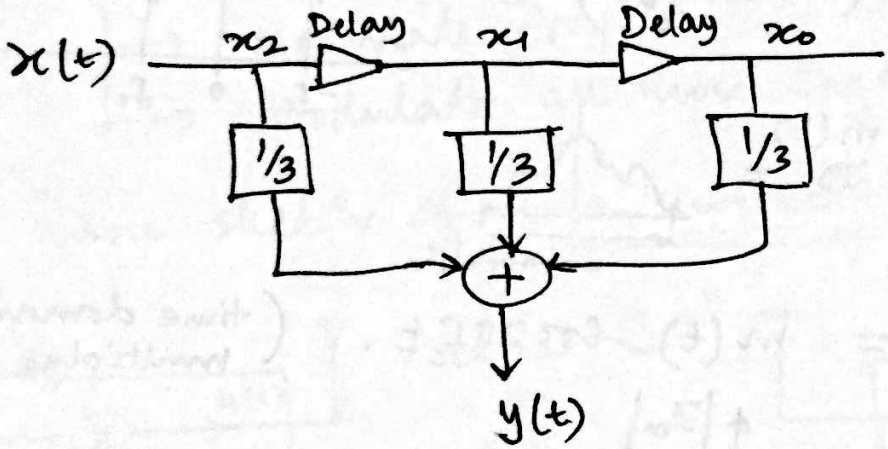


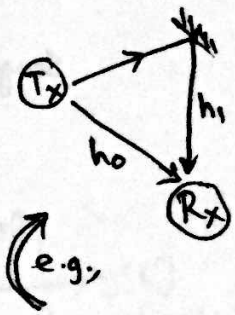
$y_0 =$ n/a

$y_1 =$ n/a

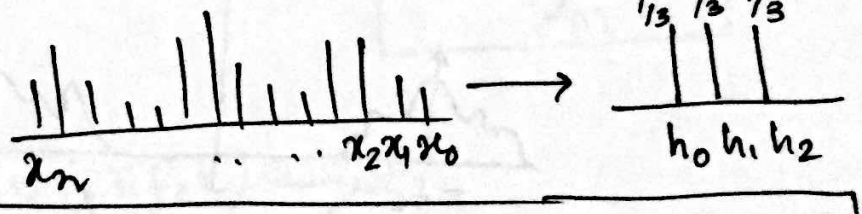
$$y_2 = \frac{x_0 + x_1 + x_2}{3}$$

$$y_3 = \frac{x_1 + x_2 + x_3}{3}$$





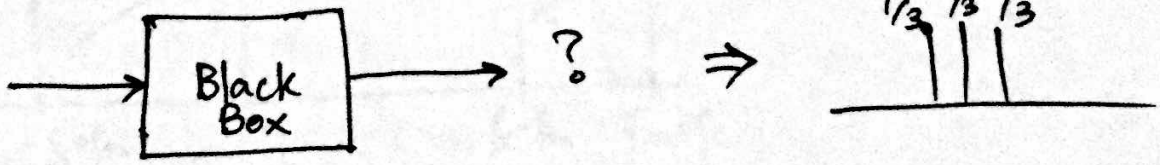
Think of this as



$$\boxed{y(n) = \sum_{\tau=0}^{2} h(\tau)\, x(n-\tau) \Rightarrow y = h * x}$$

⊛ ~~Duality of~~

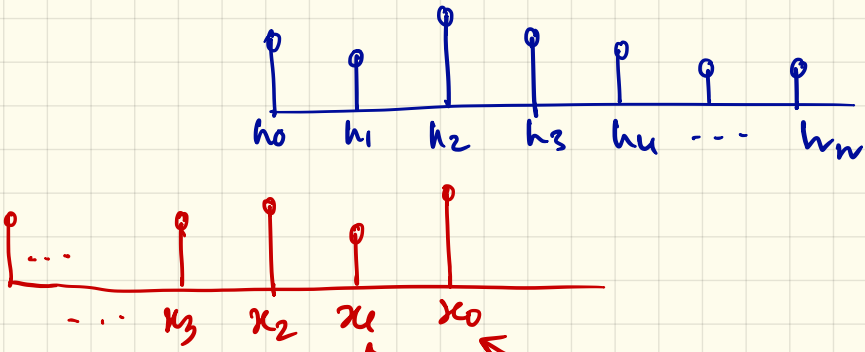
unit impulse



i.e., the ~~channel~~ impulse response of the system.

Convolution Steps.

1. Take a signal and flip it by x axis
2. Multiply and add all the values
3. Shift the signal step by step, and repeat multiply & add.



$$y_n = x_n h_0 + x_{n-1} h_1 + \dots x_0 h_2$$

Convolution:

$$\boxed{y_n = \sum_{i=0}^{n} x(n-i)\, h(i)}$$

Multiplication & Conv.

Convolution theorem.

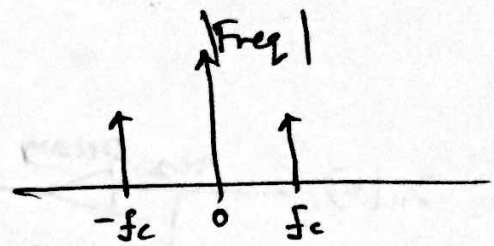
$$\text{FFT}(y_n = h_n * x_n) = H(m) \cdot X(m)$$

$$\text{IFFT}(Y(m) = H(m) \cdot X(m)) = h_n * x_n$$

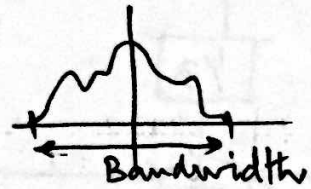
$$\text{FFT}(h_n \cdot x_n) = H(m) * X(m)$$

$$\text{IFFT}(H(m) * X(m)) = h_n \cdot x_n$$

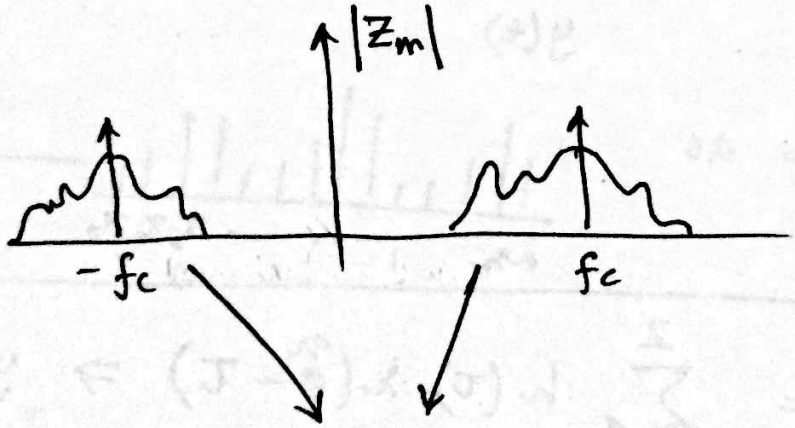
Example: $\text{FFT}(\cos 2\pi f_c t) \equiv$



say a signal $m(t)$ =

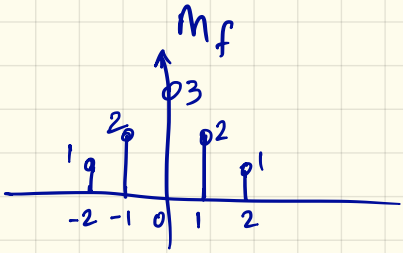


say $z(t) = m(t) \cdot \cos 2\pi f_c t$. (time domain multiply)

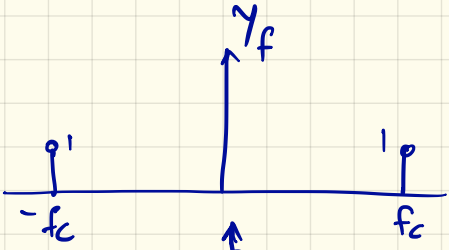


Remember, a signal convolved with an impulse signal, gives you the Original Signal.

(→) convolution in freq. domain



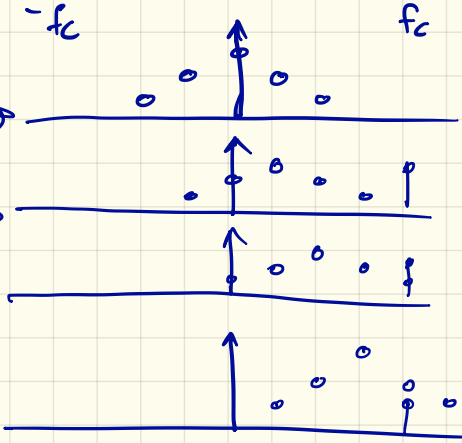
$*$



$y_0 = 0$

$y_1 = 0$

$y_2 = 1.1$



$y_3 = 1.2$

$y_4 = 1.3$

$y_5 = 1.2$

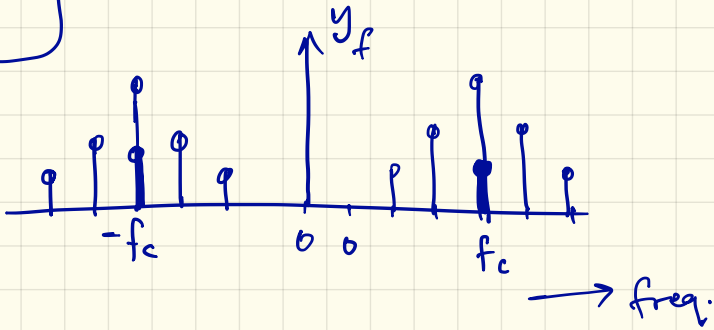
$y_6 = 1.1$

$y_7 = 0$

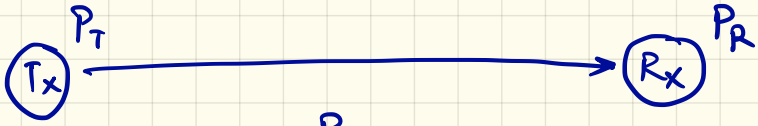
Similarly

$y_{-1} = 0 \quad y_{-3} = 1.2$

$y_{-2} = 1.1 \quad y_{-4} = 1.3 \quad \ldots$



Wireless Channel:



$$P_R = \frac{P_T}{r^{\alpha}}$$

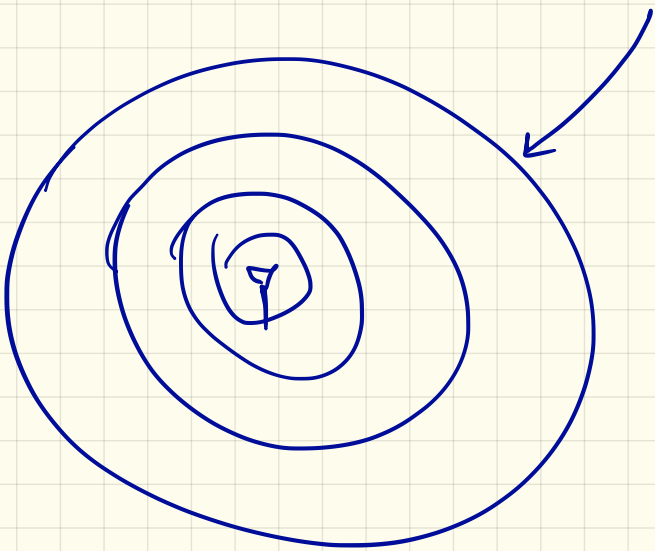
α = pathloss index

$2 \leq \alpha < 4$

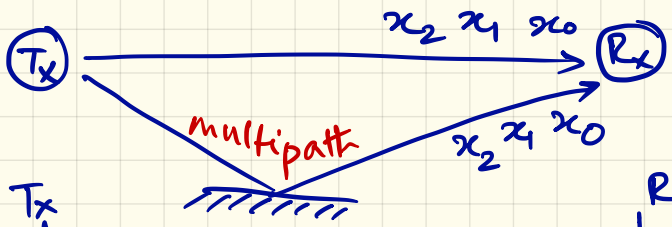
Line of Sight Channel

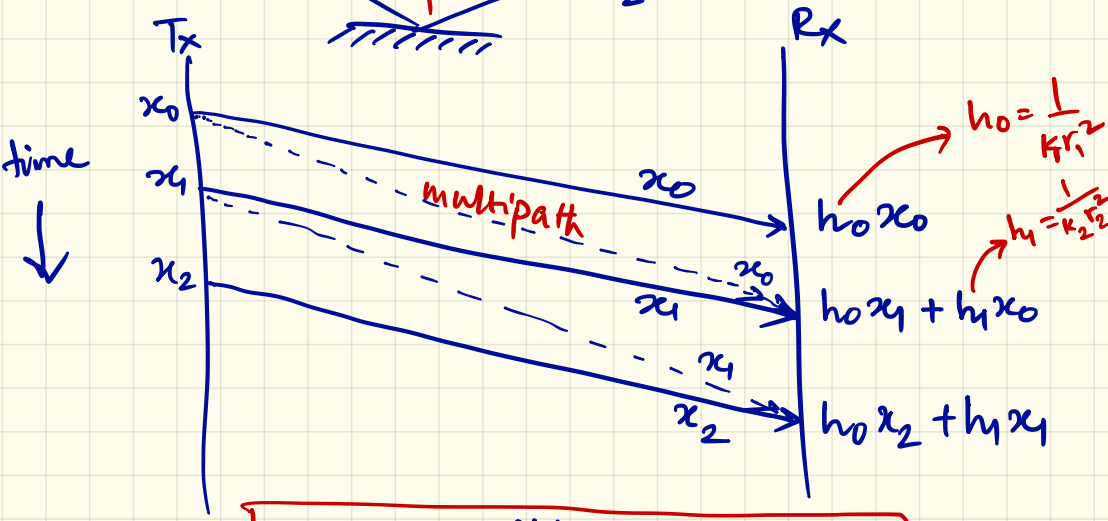
Why r^{α}? Because energy dissipates.

Area of Sphere $= 4\pi r^2$.



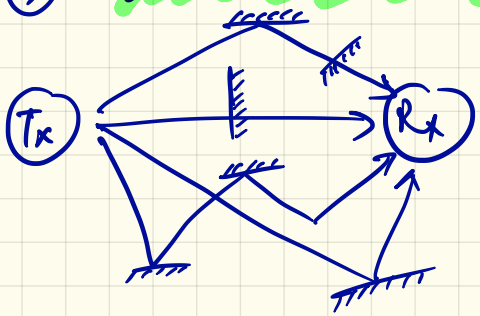
→ More realistic : Multipath channel





$$\therefore \quad \boxed{y[n] = \sum_{i=0}^{N-1} h(i)\, x(n-i)} \qquad y = h * x$$

→ Even more realistic :

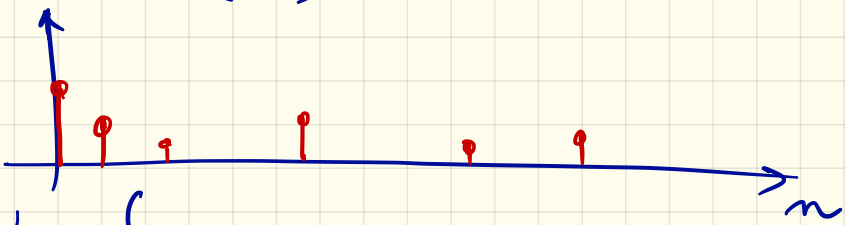


$$y[n] = h[n] * x[n] + \text{Noise}$$

hardware noise
(Random process)

③ So what does the channel look like in time domain?

$$\text{IFFT}(H_f) = h[n]$$



called CIR: Channel Impulse Response.

H_f : Called CFR: Channel Freq. Response

Q. This is like saying, if I transmitted a unit pulse from Tx, what would I receive at Rx.

Ans: The CIR.

Q: Can $h[5]$ be greater than $h[j]$, $j < 5$?
Can $h[5]$ be greater than $h[0]$?

Ans: Yes for both since attenuation also a fn. of reflecting material, & $h[0]$ may be passing through a wall.

⊖ This means, channel also a signal $h[n]$

R_x receives $y[n]$ which is a convolution of the transmitted and channel signal.

⊖ Now recall $FFT(x_n * h_n) = X_f . H_f$

$\therefore \quad Y_f = H_f X_f$ for multipath channel.

with noise.

$$Y_f = H_f X_f + N_f$$

or $$Y_f - N_f = H_f X_f = X_f H_f$$

$$\therefore \quad \underbrace{X_f}_{\text{known preamble}} \underbrace{H_f}_{\text{unknown channel}} = \underbrace{Y_f - N_f}_{\text{Measured value}}$$

$A\vec{x} = \vec{b}$ but $\vec{b}$ not in col. space of A

$\therefore$ Apply projection / regression and solve for H_f.

→ Of course, note that $X_f H_f$ is not a dot product, instead its an element by element multiplication that gives you a vector.

i.e.,

$$X_{f_1} \cdot H_{f_1} = Y_{f_1}$$
$$X_{f_2} \cdot H_{f_2} = Y_{f_2}$$
$$X_{f_3} \cdot H_{f_3} = Y_{f_3}$$
$$\vdots$$

$$\therefore \quad X_{f_1} = \frac{Y_{f_1}}{H_{f_1}} = \frac{e^{j\theta}}{e^{j\phi}} = e^{j(\theta-\phi)}$$

→ To write this as a dot product, we can do

$$X_f = \begin{bmatrix} X_{f_1} & 0 & 0 & \cdots & 0 \\ 0 & X_{f_2} & 0 & \cdots & 0 \\ 0 & 0 & X_{f_3} & & 0 \\ 0 & 0 & 0 & & \vdots \\ \vdots & \vdots & \vdots & & X_{f_n} \end{bmatrix}$$

Then,

$$X_f \cdot H_{f_1} = Y_{f_1}$$

→ You see only where you shine light.

$$Y_f = H_f X_f + N \Rightarrow \hat{H}_f = \frac{Y_f}{X_f}$$

This X_f is like shinining light in a freq. band covered by the X_f signal.

∴ Estimated $\hat{H}_f$ is also only in that band.

→ You do not get to see the channel H_f in other freq. bands on which you have not shone any light.

- What happens when channel is changing?

$$y_t = h_0 x_t + h_1 x_{t-1} + h_2 x_{t-2} \cdots$$

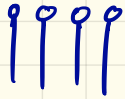
$$y_{t+1} = h_0 x_{t+1} + h_1 x_t + h_2 x_{t-1} \cdots$$

$$y_{t+2} = h_0' x_{t+2} + h_1 x_{t+1} + h_2 x_t \cdots$$

If you know $\vec{x}_t$ then you estimate the channel at every time instant.

For blind estimation, ~~I go~~ you guess $[h_0\ h_1\ h_2 \cdots h_n]$ and then ....(??)

- Difference



vs.

- fading vs. fast fading channel.

en reflected signal is very near LOS vs. very far, what does H_f look like?

- Combination of different echos of ~~xxxx~~ a signal cannot produce a freq. not present in the signal.