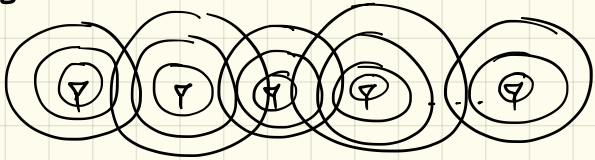
Beamforming

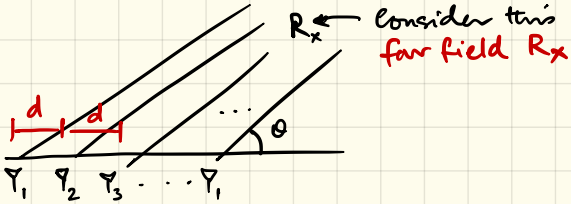
- Array processing basics
- Beam steering
- Delay sum
- Angle of arrival (AoA)
- GCC-PHAT
- MUSIC
- Blind channel inference

Basic Beamforming

① many antennas sending omni, simultaneously.



②



③ Transmit $x(t)$

Received from antenna # 1 = $s_1(t) = h(t) * x(t)$

Actual signal received = $y(t) = \sum_{i=1}^{N} s_i(t)$

④ Now,

$s_2(t)$ travels $d\cos\theta$ distance less

$\therefore$ Phase difference between $s_1(t)$ & $s_2(t)$

$$= \frac{d\cos\theta}{\lambda} \cdot 2\pi$$

$$\therefore \quad s_2(t) = s_1(t)\, e^{-j\frac{d\cos\theta}{\lambda}\cdot 2\pi} = s_1(t)\, e^{-j\phi}$$

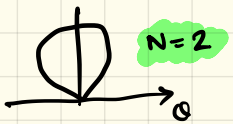
$$\therefore \quad s_i(t) = s_1(t)\, e^{-j(i\phi)}$$

Note: This is only true for a single freq. phasor. For any real or wide band signal, phase shift needs to be correctly added to each freq. component.

$$\therefore \quad y(t) = \sum_{i=1}^{N} s_i(t) = s_1(t) \sum_{i=0}^{N-1} e^{-j(i-1)\phi}$$

$$y(t) \cong \mathrm{Re}\left\{ s_1(t)\, \frac{1 - e^{-jN\phi}}{1 - e^{j\phi}} \right\}$$

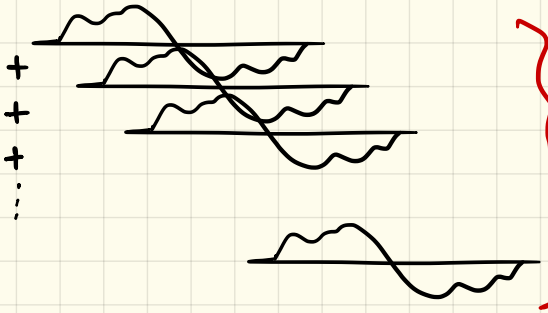
⑤ Now plot $y(t)$ for diff values of θ



Beam Rotation

- Now I want the main lobe to point towards my intended Rx at angle ϕ.
- Since $y(t) = s_1(t) + s_1(t)e^{-j\phi} + s_2(t)e^{-j2\phi} \ldots$

This is like



- For max SNR at Rx, shift each $s_i(t)$ to compensate the path delay.
- Let's transmit $s_i(t)$ as $s_1(t)e^{j(i\beta)}$

$$\therefore \quad y(t) = s_1(t) \sum_{i=0}^{N-1} e^{-j(i-1)\phi + j(i-1)\beta}$$

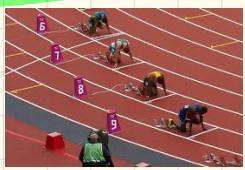
$$= s_1(t) \sum_{i=0}^{N-1} e^{-j(i-1)(\phi-\beta)}$$

To maximize $y(t)$, we make $\phi = \beta$.

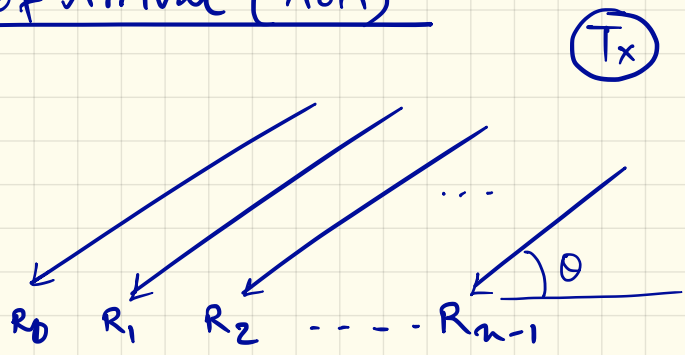
$$\therefore \quad y(t) = N s_1(t)$$

and each transmitted $\quad s_i(t) = x_1(t)e^{j(i-1)\phi}$

- This is called **Delay Sum Beamforming.**



⑦ Angle of Arrival (AoA)



$$\begin{bmatrix} y_{R0} \\ y_{R1} \\ y_{R2} \\ \vdots \\ y_{Rn-1} \end{bmatrix} = \begin{bmatrix} \cos 2\pi ft \\ \cos 2\pi ft + \phi \\ \cos 2\pi ft + 2\phi \\ \vdots \\ \cos 2\pi ft + (n-1)\phi \end{bmatrix} = e^{j2\pi ft} \begin{bmatrix} 1 \\ e^{j\phi} \\ e^{j2\phi} \\ \vdots \\ e^{j(n-1)\phi} \end{bmatrix}$$

$$\begin{bmatrix} y_{R0} \\ y_{R1} \\ \vdots \\ y_{Rn-1} \end{bmatrix} = \begin{bmatrix} 1 \\ e^{j\phi} \\ e^{j2\phi} \\ \vdots \\ e^{j(n-1)\phi} \end{bmatrix} x(t)$$

AoA estimate:

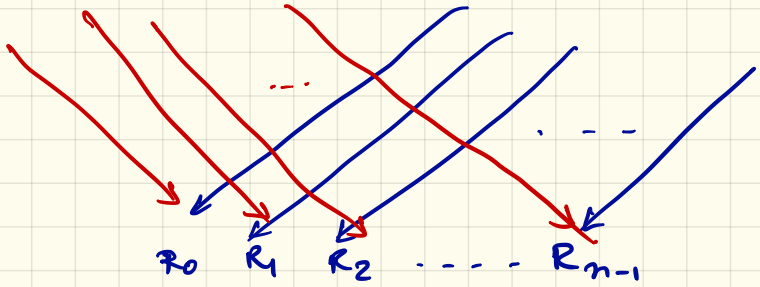
for $\theta_i = 1$ to 360

$\phi_i = f(\theta_i)$

Correlate for ϕ_i

Plot AoA spectrum

For multiple sources:



$$\begin{bmatrix} y_{R_1} \\ y_{R_2} \\ \vdots \\ y_{R_{n-1}} \end{bmatrix} = \begin{bmatrix} e^{0} & e^{0} \\ e^{j\phi_1} & e^{j\phi_2} \\ e^{j2\phi_1} & e^{j2\phi_2} \\ \vdots & \vdots \\ e^{j(n-1)\phi_1} & e^{j(n-1)\phi_2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Array output — Steering matrix (M) — Source signals.

→ So how do you pull out individual signals?

$$\begin{bmatrix} e^{0} & e^{j\phi_1} & e^{j2\phi_1} & \cdots & e^{j(n-1)\phi_1} \end{bmatrix}^{*} \begin{bmatrix} y_{R_1} \\ y_{R_2} \\ \vdots \\ y_{R_{n-1}} \end{bmatrix} = \begin{bmatrix} e^{0} & e^{j\phi_1} & e^{j2\phi_1} & \cdots & e^{j(n-1)\phi_1} \end{bmatrix}^{*} . M . \vec{x}$$

= Large value if AOA is along ϕ_1

$$\begin{bmatrix} e^0 & e^{j\phi_1} & e^{j2\phi_1} & \cdots & e^{j(n-1)\phi_1} \end{bmatrix}^* \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$= \begin{bmatrix} e^0 & e^{j\phi_1} & \cdots & e^{j(n-1)\phi_1} \end{bmatrix}^* \begin{bmatrix} e^0 & e^0 \\ e^{j\phi_1} & e^{j\phi_2} \\ \vdots & \vdots \\ e^{j(n-1)\phi_1} & e^{j(n-1)\phi_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$+ \begin{bmatrix} e^0 & e^{j\phi_1} & \cdots & e^{j(n-1)\phi_1} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_n \end{bmatrix}$$

noise

Let's denote this V_{ϕ_1}.

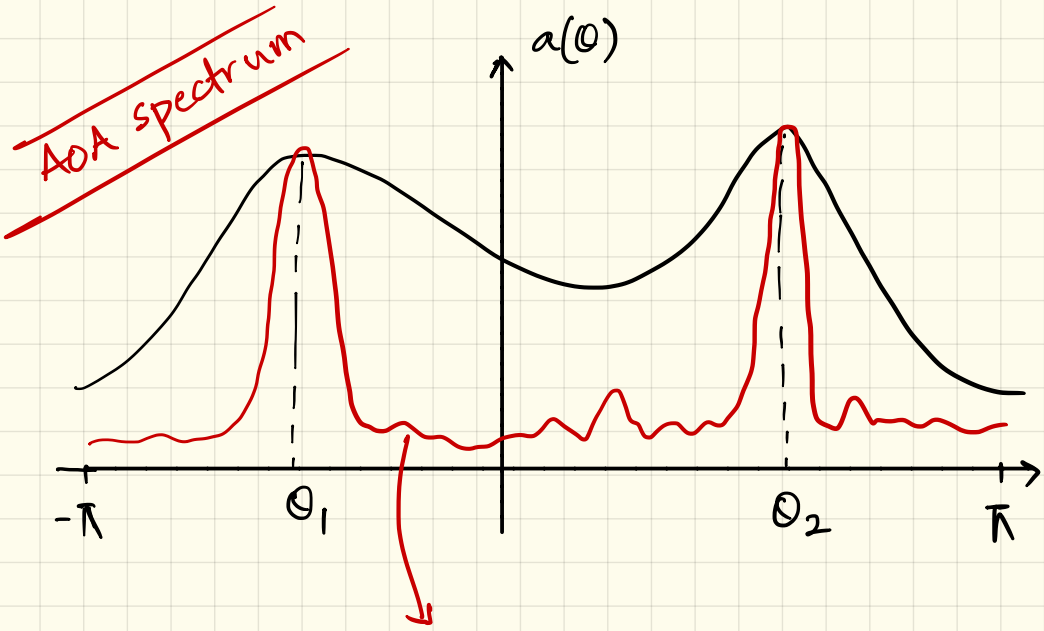
Thus, when ϕ_1 matches with actual $\phi_1 = \frac{d\cos\theta_1}{\lambda} \cdot 2\pi$

then,

$$\begin{bmatrix} — & V_{\phi_1} & — \end{bmatrix}^* \begin{bmatrix} | & | \\ V_{\phi_1} & V_{\phi_2} \\ | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -V_{\phi_1}- \end{bmatrix}^* \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_n \end{bmatrix}$$

$$= \underbrace{\|V_{\phi_1}\|^2 x_1}_{\text{Positive}} + \underbrace{\vec{V}_{\phi_1} \cdot \vec{V}_{\phi_2} x_2}_{\text{Hopefully small}} + \underbrace{\vec{V}_{\phi_1} \cdot \vec{n}}_{\text{Hopefully Small}}$$

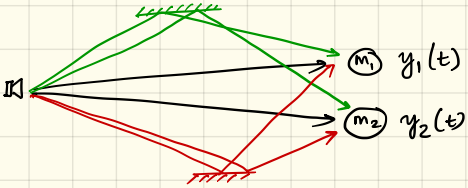
By steering along all directions θ, we get AoA spectrum, $a(\theta)$



Peaks sharpen when

- Different sources are uncorrelated
- signal itself has good autocorrelation
- Long enough signal so that it's uncorrelated to noise.

Generalized Cross Correlation (GCC):



The GCC algorithm:

GCC psendo code:

Input: $y_1(t)$, $y_2(t)$

1. $Y_1(f) = FFT(y_1(t))$
2. $Y_2(f) = FFT(y_2(t))$
3. For increasing values of τ

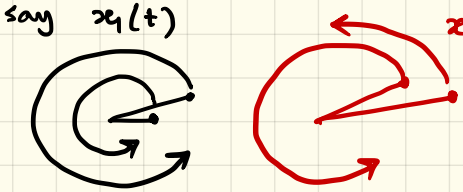
$$G(\tau) = \sum_{f=0}^{N} \frac{Y_1(f) Y_2(f) e^{j2\pi f\tau}}{|Y_1(f)\, Y_2^*(f)|}$$

What's the meaning of this?

4. $g(t) = IFFT(G(\tau))$

End.

Correlation trying to derive delay difference, but the multiplication gets biased by amplitude.



You don't want their amplitudes to bias the multiplication. So get to freq. domain, then normalize by the product of amplitudes of sticks.

MUSIC :

① $\vec{X}_{m\times 1}$, 'd' → MUSIC → AoA spectrum

② $\bar{X}_{M\times 1} = [\,x_1 \quad x_2 \quad x_3 \quad \cdots \quad x_m\,]^T$ (1, 2, 3, ..., m)

③ Model : $X_{m\times 1} = A_{m\times d} \cdot S_{d\times 1} + n_{m\times 1}$

④ $A_{m\times d} = \begin{bmatrix} | & | & & | \\ A_{\theta_1} & A_{\theta_2} & \cdots & A_{\theta_d} \\ | & | & & | \end{bmatrix}$

where $A_{\theta_i} = \begin{bmatrix} 1 \\ e^{j\frac{2\pi}{\lambda} d\cos\theta_i} \\ e^{j\frac{2\pi}{\lambda} \cdot 2d\cos\theta_i} \\ \vdots \\ e^{j\frac{2\pi}{\lambda}(m-1)d\cos\theta_i} \end{bmatrix}$

and $S_{d\times 1} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_d \end{bmatrix}$ s_i = source signal

⑤ $n_{m\times 1} = \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_m \end{bmatrix}$ n_i is noise at antenna i

auto correlation

$$\textcircled{1}\quad E[xx^H] = E[(A.s + n)(A.s + n)^H]$$

$$= E[(A.s + n)(s^H A^H + n^H)]$$

$$= E[A\,ss^H A^H + A.s.n^H + ns^H A^H + nn^H]$$

$$= E[A\,ss^H A^H] + 0 + 0 + E[nn^H]$$

$$\underbrace{R_{xx}}_{m\times m} = \underbrace{A.R_{ss}A^H}_{A_{m\times d}.R_{ss_{d\times d}}A^H_{d\times m}} + \underbrace{\sigma^2 I}_{\sigma^2.I_{m\times m}}$$

This matrix is $Q_{m\times m}$ and $\text{Rank}(Q) = d$

Rank $(I) = m$

$$\textcircled{2}\quad \det(A.R_{ss}.A^H) = 0$$

since $Q_{m\times m}$ has rank $d < m$.

$Q_{m\times m}$ is positive semi definite (i.e., all non-negative eigen values)

$$\therefore \det(R_{xx} - \sigma^2 I) = 0 = \det(AR_{ss}A^H)$$

This implies : σ^2 is a eigen value.

Also, This is minimum eigen value.

because R_{xx} has m eigen values because of noise, but m-d must be ideally 0, but here they are σ^2.

Now, $R_{xx}.e_i = \lambda_i I e_i \Rightarrow$ Eeigen Eq^n.

$$\therefore (AR_{ss}A^H + \sigma^2 I)e_i = \lambda_i I e_i$$

For minimum $\lambda_i = \lambda_{min}$,

$$(AR_{ss}A^H + \lambda_{min} I)e_{min} = \lambda_{min} I e_{min} \Rightarrow AR_{ss}A^H e_{min} = 0$$

Since R_{ss} is rank d, and A has no null space,

$$\therefore \quad A^H \cdot e_{min} = 0$$

i.e., e_{min} vectors $\perp$ to the mixing matrix columns.

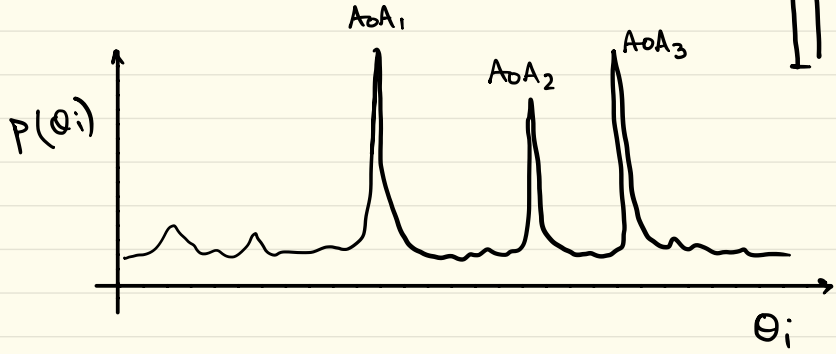
$\therefore$ For the correct A_{θ_i}, $A^H_{\theta_i} \cdot e_{min} = 0$

For wrong A_{θ_i}, $A^H_{\theta_i} e_{min}$ should have some non-zero value

$\therefore$ AoA spectrum $P(\theta_i) = \dfrac{1}{\| A^H_{\theta_i} \cdot e_{noise} \|}$

$1 \times m$ $\quad$ $m \times (m-d)$

$$\left[\text{———} \right] \left[\begin{matrix} | & | & | \\ | & | & | \end{matrix} \right] = \left[0 \; 0 \right]$$



⊙ Questions around MUSIC

Q① If sources are not independent (e.g., multipath) where will the math break?

Q② If $d > M$, why does the math break?

Q③ Why is the A matrix independent?

A③ Vandermond matrices are of the form

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & & \alpha_d \\ \alpha_1^2 & \alpha_2^2 & & \alpha_d^2 \\ \alpha_1^3 & \alpha_2^3 & & \alpha_d^3 \\ \vdots & \vdots & & \vdots \\ \alpha_1^M & \alpha_2^M & & \alpha_d^M \end{bmatrix}$$

are independent, so long as $d < m$.

The proof is that the $\det\left(\text{Vandermond matrix}\right) = \prod(\alpha_i - \alpha_j)$

which is only 0 only when $\alpha_i = \alpha_j$

Blind Channel Estimation

- Consider 2 microphones m_1 and m_2.

- At each mic. we record:

$$Y_1 = H_{room} \cdot X$$

$$Y_2 = H_{room2} \cdot X$$

We do not know x, and we do not know any h_{room}.

- Question is: Can we get a better version of what the user is saying, i.e., some estimate of $\hat{x}$ that is better than y.

- Observe that $\frac{Y_1}{H_1} = \frac{Y_2}{H_2}$

$$\therefore \quad Y_1 H_2 - Y_2 H_1 = 0$$

Let's also assume H_1, H_2 are sparse, i.e., L_1 norm of H_1, H_2 are small. (Recall, L_1 norm counts no. of non-zero elements.)
L_0 norm is better but NP complete.

Thus, now we formulate an objective fn. as

$$\text{minimize} \quad Y_1 H_2 - Y_2 H_1 + \lambda\left(\|H_1\|_1 + \|H_2\|_1\right)$$

$$\text{s.t.} \quad \|H\|_2^2 = k \quad , \quad \text{say } k=1$$

where $\|H\| = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$

- $H_1 = H_2 = 0$ is minimum but infeasible since $\|H\|_2^2 = 1$.

- The λ regularization adds a penalty for $\|H_1\|_1$ and $\|H_2\|_1$ becoming larger.

  Ideally, we should have done L_0 norm, which would reduce the # of non-zero elements in H_1 and H_2.