DSP:
Fourier Transform
Model multiplicity $\rightarrow$ count/numbers.

Model opposite $\rightarrow$ I have $5$

How much should I add to make $0$? Ans: $-5$

Numbers & neg. numbers are all abstractions of reality.

Now, how to model rotation?

We have $\cos\theta$ and $\sin\theta$.

Of course we need to express their relation in some way. We cannot just say $\cos\theta + \sin\theta \times$

or $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

Which still does not allow algebra.

Ok, we can say $x \cos\theta + y \sin\theta$

But we don’t have to say $x^2$, we just need to make $y^2$ become $1$ to $x^2$.

How? Well, one way is to say that $y^2$ should be $-1$.

i.e., $y^2 = -1$ \therefore $y = \sqrt{-1} = j$

$\therefore$ Rotation is a vector of $\cos\theta + j\sin\theta$

Aha, this is exactly $e^{j\theta}$ $\Rightarrow$ models rotation.
Fourier observed each column in orthogonal.

\[
\begin{align*}
\text{freq 0} & : \begin{bmatrix} e^{i0} \end{bmatrix} \\
\text{freq } f_1 & : \begin{bmatrix} e^{i0 \cdot \frac{2\pi}{N}} & e^{i1 \cdot \frac{2\pi}{N}} \end{bmatrix} \\
\text{freq } 2f_1 & : \begin{bmatrix} e^{i0 \cdot \frac{4\pi}{N}} & e^{i1 \cdot \frac{4\pi}{N}} \end{bmatrix} \\
\text{freq } mf, & \begin{bmatrix} e^{i0 \cdot \frac{m2\pi}{N}} & e^{i1 \cdot \frac{m2\pi}{N}} \end{bmatrix}
\end{align*}
\]

You can immediately see because

\[
\begin{bmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
6 & 5 & 4 & 3 & 2 & 1 & 0
\end{bmatrix}
\]

Same holds for any pair.
Vector Projection to Fourier Basis

\[ \vec{x} = \begin{bmatrix} x[0], x[1], x[2], \ldots, x[N-1] \end{bmatrix} \]

Vector in N-dimensional space.

\[ \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} \]

Basis in time domain.
Let's denote $x[0]$ as $x_0$ from now on.

Discrete Fourier Transform (DFT)

$$\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_0 \\
x_1 \\
x_2 \\
\vdots \\
x_{N-1} \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
f_0 & f_1 & f_2 & \cdots & f_{N-1} \\
\end{bmatrix}
\begin{bmatrix}
z_0 \\
z_1 \\
z_2 \\
\vdots \\
z_{N-1} \\
\end{bmatrix}
$$

Orthogonal Fourier Basis matrix

$z(m) = F^{-1} \cdot x$

$z(m) = (F^*)^T \cdot x$

$\begin{bmatrix}
z_0 \\
z_1 \\
z_2 \\
\vdots \\
z_{N-1} \\
\end{bmatrix} =
\begin{bmatrix}
f_0^* & f_1^* & f_2^* & \cdots & f_{N-1}^* \\
\end{bmatrix}
\begin{bmatrix}
x_0 \\
x_1 \\
x_2 \\
\vdots \\
x_{N-1} \\
\end{bmatrix}$
DFT Equations

Consider 3D case: \[
\begin{bmatrix}
\bar{z}_0 \\
\bar{z}_1 \\
\bar{z}_2
\end{bmatrix}
= 
\begin{bmatrix}
\bar{f}_0^* \\
\bar{f}_1^* \\
\bar{f}_2^*
\end{bmatrix}
\begin{bmatrix}
x_0 \\
x_1 \\
x_2
\end{bmatrix}
\]

\[
\bar{z}_0 = e^{-j\frac{2\pi}{N}0.0}x_0 + e^{-j\frac{2\pi}{N}0.1}x_1 + e^{-j\frac{2\pi}{N}0.2}x_2
\]

\[
\bar{z}_1 = e^{-j\frac{2\pi}{N}1.0}x_0 + e^{-j\frac{2\pi}{N}1.1}x_1 + e^{-j\frac{2\pi}{N}1.2}x_2
\]

\[
\bar{z}_2 = e^{-j\frac{2\pi}{N}2.0}x_0 + e^{-j\frac{2\pi}{N}2.1}x_1 + e^{-j\frac{2\pi}{N}2.2}x_2
\]

DFT:
\[
\bar{Z}(m) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}m.n}
\]

IDFT:
\[
x[n] = \sum_{m=0}^{N-1} \bar{Z}(m) e^{j\frac{2\pi}{N}m.n}
\]

Scaling by \(\frac{1}{\sqrt{N}}\)

Note that dot product = Projection only when the dot product is normalized.

Now, for DFT, observe that each column of \(F\) is not a unit vector, e.g., \([1\ 1\ 1]^T\).

- The DFT needs to be scaled by the length of the vector \(\sqrt{N}\).

i.e., DFT \(X_m = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}m.n}\)

IDFT \(x_n = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} X_m e^{j\frac{2\pi}{N}m.n}\)
Apply Parseval's synthesis identity

\[ x[n] = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} Z(m) e^{j \frac{2\pi}{N} mn} \]

Add up the projections multiplied by the corresponding basis vector.

\[ x[1] = Z(1) e^{\frac{j 2\pi}{N} . 1 . n} + Z(2) e^{\frac{j 2\pi}{N} . 2 . n} + Z(3) e^{\frac{j 2\pi}{N} . 3 . n} \]
Analogy:

Tomato $x \rightarrow Pizza$

Pasta

- cook Italian food with 2 different "Basis ingredients"

DFT helps go from one basis to another.
$Z_m$ is a complex vector

- **Magnitude ($|Z_m|$)**
- **DC or avg. of signal**
- **Phase ($\angle Z_m$)**

Max freq. of a given signal = BANDWIDTH

Translating to real-world frequencies:

- Say sampling freq. = $f_s$
- Now, slowest freq. = $N$ samples/cycle
- $N$ samples take $N \cdot \frac{1}{f_s}$ time
- 1 cycle takes $\frac{N}{f_s}$ time
- Slowest freq. = $\frac{f_s}{N}$ Hz.
Increasing freq. @ 0, $\frac{fs}{N}$, $\frac{2fs}{N}$, $\frac{3fs}{N}$ \ldots $\frac{(N-1)fs}{N}$

Thus, when analyzing a given signal, we have two knobs:
- Sampling freq. $fs$
- No. of FFT points $N$

Important to note:
1. Large $fs$ means we can see till large freq. component $fs$
2. Large $N$ implies we have better resolution of $fs$ (i.e., $fs/N$)
Everyday signal $\cos 2\pi f_m t$

Discrete sampled signal $x[n] = \cos 2\pi f_m n t_s$

where $t_s = \frac{1}{f_s}$ is sampling interval, $f_s$ is sampling freq.

Now what is the DFT ($x[n] = \cos 2\pi f_m n t_s$)

Real signals always symmetric because the imaginary component needs to be cancelled out by a much faster moving rotation. Also, phase ZERO because all rotations start from zero.
What is DFT ($x[n] = e^{j2\pi f_1 n}$)?

Just one stick rotating at $f_1$ is good enough to create this signal.

Now, what is DFT ($x[n] = \sin(2\pi f_1 n)$)?
When we took the DFT of the \( \cos(2\pi f_0 t) \) signal, the implicit assumption is that we took \( N \) points that exactly covers integer no. of \( \cos(t) \) cycles.

If we take \( N \) samples that only covers part of the signal, say

Then, we can no longer form this signal with just the \( f_0 \) freq. component.

Instead, this signal \( Y \) can be viewed as a truncated \( X \) signal, truncated by a pulse train of \( N \) samples.

\[
\text{DFT}(Y) = \text{DFT}(X \cdot \text{PulseTrain})
\]

\[
\neq \text{DFT}(X) \quad \text{so be careful.}
\]
Properties of DFT

1. DFT is linear:
\[ \text{DFT} (x[n] + y[n]) = \text{DFT}(x[n]) + \text{DFT}(y[n]) = X_m + Y_m \]

2. For real signals, DFT symmetric (Negative freq.)

Intuitively, the second stick is rotating in the clockwise direction to cancel the imaginary parts of the real signal.

Real signals have symmetric spectrogram because imaginary parts cancelled by faster moving stick.

So this is equivalent to taking the mirror image and pasting it on the negative freq. axis.
DFT of shifted signal is original DFT with just a phase shift.

\[ X'[n] = x[n+k] \]

i.e., \( X'[n] \) is shifted by \( k \) samples. Then

\[ Y[n] = x[n+k] \]

\[ X_m = \sum_{n=0}^{N-1} x[n+k] e^{-j\frac{2\pi}{N}m.n} = \sum_{h=0}^{N-1} x[n+k] e^{-j\frac{2\pi}{N}(n+k)\frac{mk}{N}} \]

\[ Y_m = e^{j\frac{2\pi}{N}m.k} X_m \]

means a phase shift only to \( X_m \).

\[ X'_m = e^{j\frac{2\pi}{N}k.m} X_m \]

means only phase shift
Real Spectrogram

Voice Signal.

Shows how a specific freq. comes and goes over time.

Shows various freq. components at this time instant.
Nyquist's Sampling

Say you are given the following samples and asked to reconstruct the analog signal... and say values match $\sin(\cdot)$

We can say freq $f = 10$ Hz

$\therefore$ Analog signal $= \sin(2\pi f n t s)$

But is this the only signal that fits these values?

Observe: $\sin(2\pi f n t s + 2\pi m)$ also fits the values given, if $m = \text{integer}$

i.e., $\sin(2\pi (f_0 + \frac{m}{n t s}) n t s)$ fits values

Choosing $m$ as multiple of $n$ ($\frac{m}{n} = k$)

$\sin(2\pi (f_0 + k \cdot \frac{1}{t s}) n t s)$

$= \sin(2\pi (f_0 + k f s) n t s)$ \ldots fs \text{ i.e. sampling freq.}$
This means, many other frequencies, separated by kfs from the core freq. f, fits the given set of points.

\[ \downarrow \]

Called "ALIASED" frequencies.

ie, \((f + kfs)\) are aliases of f for different values of k.

\[ \downarrow \]

If I sample a signal at fs, and therefore get to see freq. components of \(\frac{fs}{N}, \frac{2fs}{N}, \frac{3fs}{N}, \ldots, \frac{N-1fs}{N}\),

actually, each of these freq. components could have aliased freq. from other freq. contained in the signal.

To ensure we remove aliasing (which pollutes), we need to understand max freq. component of signal, say \(fw\).
Observe then higher the \( f_s \), more separated are the aliased freq. Example

\[
\begin{align*}
&f-2f_s \quad f-f_s \quad f \quad f+f_s \quad f+2f_s \quad f+3f_s \quad \ldots
\end{align*}
\]

Now, to avoid pollution for a given signal of bandwidth \( f_w \), we need no freq. within it to alias to within its bandwidth.

That is \( f_w \) should alias to left of \(-f_w\) and \(-f_w\) should alias to right of \( f_w \)

This implies:

\[
\begin{align*}
f_w - f_s &< -f_w \\
\text{or} \quad -f_w + f_s &> f_w
\end{align*}
\]

\[
\downarrow
\]

\( f_s > 2f_w \)

i.e., sampling freq. must be twice b/w to have an unpolluted reconstruction of the signal.
Thus, sampling at $f_s$ may show the alias of $(-fw + fs)$.

However, knowing $fw$, we can filter out the signal at $fw$, thereby removing all the signals outside $[-fw, fw]$ alias free, giving us a full reconstruction of the signal of interest.

\[\rightarrow \text{Low Pass filter (LPF)}\]

\[\text{alias free original signal.}\]
The signal processing flow.

1. Input signal (e.g., voice)
2. LPF (Filter at $f_w$ ($\approx 20\text{kHz}$))
3. Nyquist sample
4. Sample at $f_s \geq 40\text{kHz}$
5. Digital samples
6. N point FFT, say $N = 1024$
7. Visualize

$X_t$, $X[n]$, $|Z_m|$, $X(zm)$
Questions?