→ Vector Space:

$\mathbb{R}^2$ = all 2D vectors = xy plane

Space → • $\vec{u} + \vec{v}$ falls in same space
→ $c\vec{u}$ also falls in same space

→ Quiz: Subspace or not?

(1) 2D plane?
(2) one quadrant?
(3) Line $x = y$?
(4) Zero vector?

→ Null Space:

$A\hat{x} = 0$, if $\hat{x} \neq 0$ then

$c\begin{bmatrix} | \\ x \\ | \end{bmatrix}$ forms the null space.

NULL SPACES

- $A\vec{x} = \vec{b}$

  ↳ What vector takes matrix A to zero.

  $$A . \vec{X}_{null} = 0$$

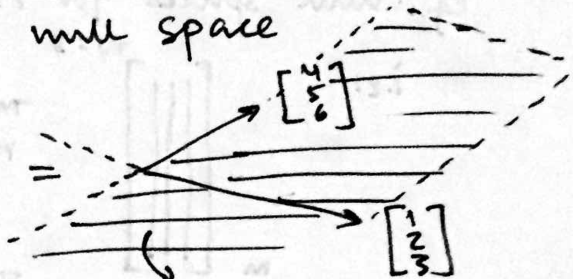
  Of $x_{null} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ but that's trivial.

  But suppose $x_{null} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$. Then $c.\vec{X}_{null}$ will also take $A \to 0$, where c is any constant.

- The space of cx_{null} is the null space

- Ex.

  $$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \Longrightarrow C(A) =$$



2D plane formed by two col. vectors.

Say $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; Then $N(A) = c\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

when talking about N(A), view the matrix from the perspective of columns.

- Now cosider row space $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$. But to talk about null space let's make them columns

  $\therefore A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Now $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

  $\therefore$ Row space $(A) = C(A^T)$

  Null space of $(AT) = N(A^T) = c\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

- Of course $N(A) \neq N(A^T)$
- However, observe that columns of A, e.g., $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is orthogonal to $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

  because $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ and we know $X^TY = 0$ implies $X \perp Y$.

→ Thus $N(A^T)$ is $\perp$ $C(A)$

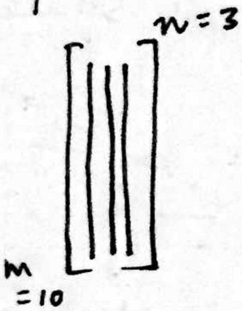
Similarly $N(A) \perp C(A^T)$

because $\begin{bmatrix}1\\4\end{bmatrix}^T \begin{bmatrix}x_1\\x_2\end{bmatrix} = 0$, $\begin{bmatrix}2\\5\end{bmatrix}^T \begin{bmatrix}x_1\\x_2\end{bmatrix} = 0$, $\begin{bmatrix}3\\6\end{bmatrix}^T \begin{bmatrix}x_1\\x_2\end{bmatrix} = 0$

→ Now what does $N(A^T) \perp C(A)$ mean?

If A is a thin matrix, then there must be lot of null spaces for A^T.

i.e.,



means 10 3D vectors are very redundant, so lot of null space for the row space of A.

This $N(A^T)$ is $\perp$ $C(A)$.

Thus $N(A^T)$ fills up the gap between $\mathbb{R}^m$ and $C(A)$.

i.e., $\boxed{\mathbb{R}^m = C(A) + N(A^T)}$

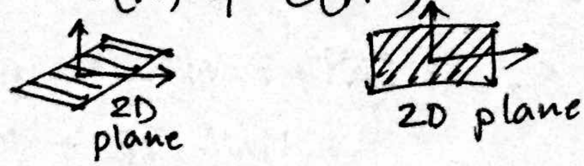
If rank of $A = r$ which is also dim(A)

then $\boxed{\dim(N(A^T)) = m - r}$

→ Similarly $\boxed{\mathbb{R}^n = C(A^T) + N(A)}$

→ Now $\dim(C(A)) = \dim(C(A^T)) = \text{rank}(A)$

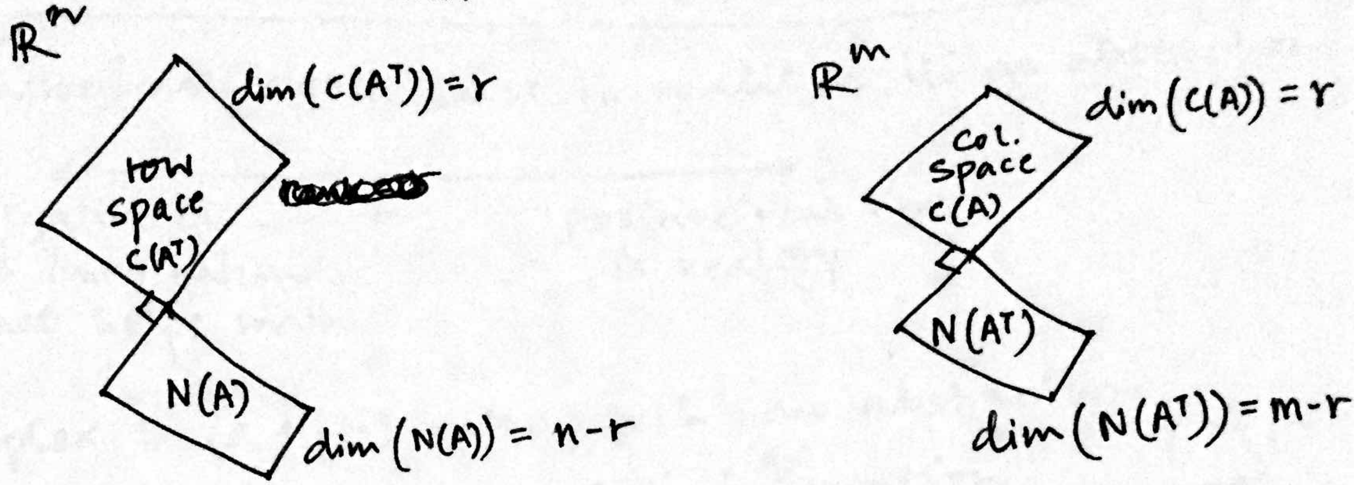
even though $C(A) \neq C(A^T)$



~~$\mathbb{R}^n = C(A^T) + N(A)$~~

$\therefore$ $\boxed{\dim(N(A)) = n - r}$

Matrix $A_{m \times n}$, rank $= r$



→ Rank

↳ No. of linearly indep. cols/rows.

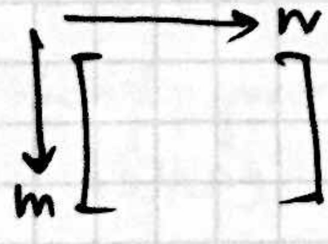
If col. vectors in A are dep., then some combination of columns will catch the other columns.

Say $A = \begin{bmatrix} | & | & | \\ c_1 & c_2 & c_3 \\ | & | & | \end{bmatrix}$

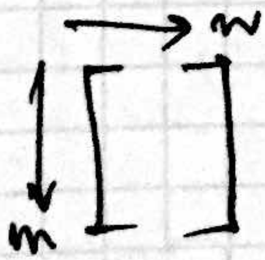
Say $w_1 c_1 + w_2 c_2 = w_3 c_3$

Then $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ is a vector in the null space.

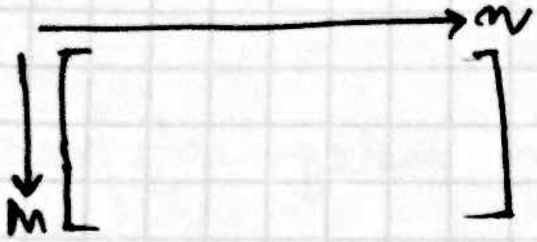
$N(A)$ = The full null space is $k \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

→ 

Rank = m = n } 3 3D vectors
Full rank
1 sol^n possible
$N(A)$ = ZERO vector



Rank = n < m } 2 3D vectors
Full col. rank
0 or 1 sol^n possible
$N(A)$ = ZERO vector



$\left.\begin{array}{l}\text{Rank} = m < n\end{array}\right\}$ 4 3D vectors

Full row rank

Null space non-zero

∞ sol^n. possible

Rank $< m$, Rank $< n$

0 or ∞ sol^{n}s.

$$\boxed{\text{Rank} \leq \min\{m, n\}}$$

→ BASIS:

Indep. vectors that span a space.

→ DIMENSION = | BASIS |

Given a space, every basis has equal # of vectors called Dimension.

Quiz: Q. dim(C(A)) = ?

A. = rank.

(5) Orthogonal Vectors:

$$x^T y = 0$$

Dot prod. $x^T \cdot y \cong$ Proj of x on y.

Quiz. Q. Wall and floor ortho?

A. No.

(3) Length $\Rightarrow$ $\|v\| = v^T v$ $\rightarrow$ L_2 norm

$\|v\|_p = \sqrt[p]{\sum_{i=1} x_i^p}$ ∴ L_0 norm is cardinality of vector, i.e., # of non zero elements

(3) uv^T = rank 1 matrix

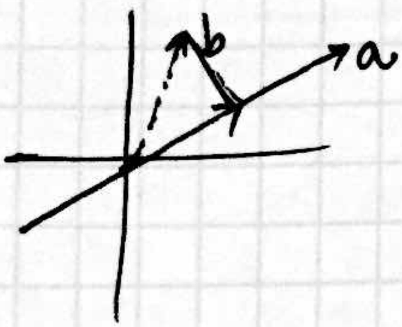
$$\begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} c & d \end{bmatrix} = \begin{bmatrix} ac & ad \\ bc & bd \end{bmatrix}$$

(5) $u^T v$ $\Rightarrow$ test for ortho

(3) $u^T u$ $\Rightarrow$ L_2 norm of u
square matrix
Symm.

3 Projection

When $\vec{b}$ not in $C(A)$, $Ax=b$ not solvable.



$$\vec{p} = x\vec{a}$$

$$\vec{p} + \vec{e} = b$$

$$\therefore \vec{e} = (\vec{b} - x\vec{a})$$

$$\vec{a} \cdot \vec{e} = a^T \cdot e = 0$$

$$\text{or} \quad a^T(\vec{b} - x\vec{a}) = 0$$

$$x = \frac{a^T b}{a^T a}$$

Projected vector $= xa = \dfrac{a^T b . a}{a^T a}$

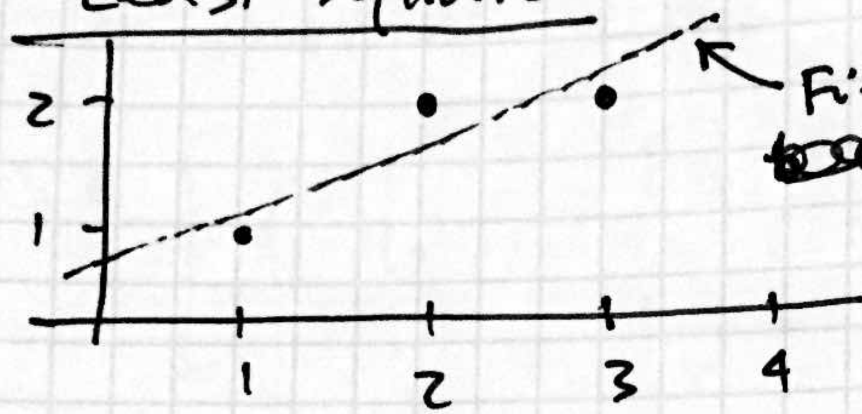
$$= \frac{a a^T b}{a^T a}$$

Proj. matrix (P) $= \left(\dfrac{a a^T}{a^T a}\right)$

$$\therefore A\hat{x} = Pb$$

$$\therefore \hat{x} = A^{-1}\left(\frac{AA^T}{A^T A}\right) b = \frac{A^T b}{(A^T A)}$$

Least Square



Fit this line

$y = mx + c$

m and c are unknowns

$x=1: \quad m + c = 1$

$x=2: \quad 2m + c = 2$

$x=3: \quad 3m + c = 2$

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}} \underset{x}{\begin{bmatrix} m \\ c \end{bmatrix}} = \underset{b}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}$$

Least Square: $\min \|Ax - b\|^2 = \|e\|^2$

Proj solves this naturally

$$\hat{x} = \frac{A^T b}{(A^T A)} = (A^T A)^{-1} \cdot (A^T b)$$

→ Orthonormal vectors

$$Q = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix}$$

$$q_i^T q_j = \begin{cases} 0 & \text{when } i \neq j \\ 1 & \text{when } i = j \end{cases}$$

If Q is square matrix

$$Q^T Q = I \quad \text{, i.e., } Q^T = Q^{-1}$$

→ Projection becomes simple.

$$\hat{x} = \frac{Q^T b}{(Q^T Q)} = Q^T b$$

→ Gram - Schmidt :

Make any matrix orthonormal